Lesson 12; part a) "Line" Integrals of Scalar Fields

"Line" Integrals, which should be called curve integrals, are used to integrate either

A. Scalar-valued functions along curves

B. Vector-valued functions along curves

In the first part of this lesson, we focus on taking line integrals of scalar-valued

$$f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$$

over a parameterized oriented curve C.

Scalar Line Integrals over $D \subseteq \mathbb{R}^2$

Let $z = f(x,y)$ be a two-variable, real-valued function

with $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Let $C_1 \subseteq D$ be a parameterized curve contained

in the domain D where

$$C_1 = \{ \vec{r}(s) \; : \; a \le s \le b \}$$

where parameter s represents the arc length of C_1

$$= \{ \langle x(s), y(s) \rangle \; : \; a \le s \le b \}$$

Let's consider the curve along the surface

$$S_{C_1} = \{ z = f(x,y) \; : \; (x,y) \in C_1 \}$$

☐ see Mma notebook fro[m] authors

$$= \{ z = f(x(s), y(s)) \; : \; a \le s \le b \}$$

Then, the area "between" curve S_C on the surface

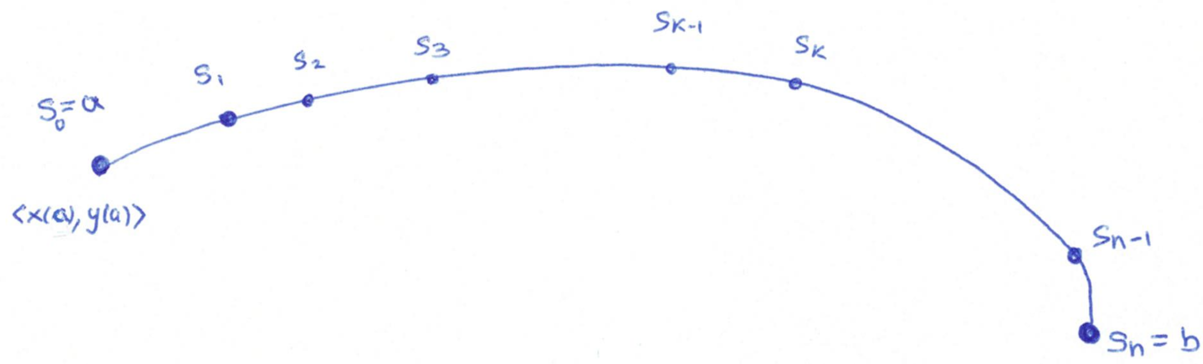
and the embedding of curve C_1 in the xy-plane

is given symbolically as

$$\int_{C_1} f \, dw = \int_{C_1} f(x(s), y(s)) \, ds$$

$$= \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x(s_k^*), y(s_k^*)) \, \Delta s_k$$

Let's visualize what's happening in our domain space:



Example 14.2.1 p. 1062

The temperature of a circular plate

$$D = \{ (x,y) : x^2 + y^2 \leq 1 \}$$

is modeled by function

$$T(x,y) = 100 \cdot (x^2 + 2y^2)$$

Find the "average" temperature on the edge of the plate.

Solution: First, let's parameterize the edge of the plate using arc length parameter. To do so, we state that the edge can be modeled as a curve C_1 with

$$C_1 = \{ \vec{r}(s) : 0 \leq s \leq 2\pi \}$$

$$= \{ \langle x(s), y(s) \rangle : 0 \leq s \leq 2\pi \}$$

$$= \{ \langle \cos(s), \sin(s) \rangle : 0 \leq s \leq 2\pi \}$$

Example 14.2.1, continued ---

The "total" temperature on the edge of the plate is given by the integral

$$\int_{C} T(x,y)\, ds = \int_{C} T(x(s), y(s))\, ds$$

$$= \int_{C} 100 \cdot (x^2 + 2y^2)\, ds$$

$$= \int_{0}^{2\pi} 100 \cdot \left((x(s))^2 + 2 \cdot (y(s))^2 \right) ds$$

$$= \int_{0}^{2\pi} 100 \cdot (\cos^2(s) + 2 \sin^2(s))\, ds$$

note: Trig Identity

$\cos^2(s) + 2\sin^2(s) = \cos^2(s) + \sin^2(s) + \sin^2(s)$

$= 1 + \sin^2 s$

$$= \int_{0}^{2\pi} 100 \cdot (1 + \sin^2(s))\, ds$$

$$\Rightarrow \int_{C_1} T \, ds = 100 \int_0^{2\pi} 1 + \sin^2(s) \, ds$$

<u>Recall: Double angle formula</u>

$$\cos(2s) = \cos(s+s)$$

$$= \cos(s) \cdot \cos(s) - \sin(s) \sin(s)$$

$$= \underbrace{\cos^2(s)} - \sin^2(s)$$

$$= (1 - \sin^2(s)) - \sin^2(s)$$

$$= 1 - 2 \sin^2(s)$$

$$\Rightarrow \sin^2(s) = \frac{1 - \cos(2s)}{2}$$

$$\Rightarrow \int_{C_1} T \, ds = 100 \int_0^{2\pi} 1 + \frac{1 - \cos(2s)}{2} \, ds$$

$$= 100 \int_0^{2\pi} \frac{3}{2} - \frac{\cos(2s)}{2} \, ds$$

$$\Rightarrow \int_{C_1} T\,ds = 100\cdot\left(\frac{3}{2}s - \frac{\sin(2s)}{4}\right)\Bigg|_0^{2\pi}$$

$$= 100\left(\frac{3}{2}\cdot 2\pi - \frac{\sin(4\pi)}{4}\right) - \left(\frac{3}{2}\cdot 0 - \frac{\sin(2\cdot 0)}{4}\right)$$

$$= 100\cdot 3\pi$$

$$= 300\pi$$

To get the average temperature, we divide by the total length of the curve

$$\Rightarrow \text{Avg}(T) = \frac{1}{L}\cdot\int_{C_1} T\,ds$$

$$= \frac{1}{2\pi}\cdot 300\pi$$

$$= \boxed{150}$$

Line Integrals of Curves with Parameters other than Arc Length

Suppose we want to find the value of a "line" integral over a curve

$$C_1 = \{ \vec{r}(t) : a \le t \le b \}$$

where the parameter t is NOT arc length.

The major idea we will use is a change of variables. Suppose

$$C_1 = \{ \vec{r}(t) : a \le t \le b \}$$

$$= \{ \langle x(t), y(t) \rangle : a \le t \le b \}$$

Recall that we could calculate the arc length of a portion of C_1 as

$$s(t) = \int_a^t \|\vec{r}\,'(u)\|_2 \, du$$

$$\Rightarrow \quad s'(t) = \frac{d}{dt}\left[s(t)\right]$$

$$= \frac{d}{dt}\left[\int_a^t \|\vec{r}\,'(u)\|_2 \, du\right]$$

$$= \|\vec{r}\,'(t)\|_2$$

$$\Rightarrow \quad ds = s'(t)\, dt = \|\vec{r}\,'(t)\|_2 \, dt$$

Remember: $ds = d[s(t)]$

$$= \underbrace{s'(t)}_{\text{"Jacobian"}}\, dt$$

$$\Rightarrow \int_{C_1} f \, ds = \int_{C_1} f(x(t), y(t)) \, ds$$

$$= \int_a^b f(x(t), y(t)) \, \|\vec{r}\,'(t)\|_2 \, dt$$

$$= \int_a^b f(x(t), y(t)) \cdot \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$

where $C_1 = \{ \vec{r}(t) : a \leq t \leq b \}$

$= \{ \langle x(t), y(t) \rangle : a \leq t \leq b \}$

$$= \int_a^b f(\vec{r}(t)) \, \|\vec{r}\,'(t)\|_2 \, dt$$

* The above derivation is focused on curves $C_1 \subseteq \mathbb{R}^2$ where the function $\vec{r} : [a,b] \longrightarrow \mathbb{R}^2$ is given as $\vec{r}(t) = \langle x(t), y(t) \rangle$. Redo this derivation to arrive at formula for arclength if $C_1 \subseteq \mathbb{R}^3$.

Example 14.2.2 p. 1063 – 1064

Confirm the average temperature computed in Example 1 where the edge of plate $D = \{(x,y) : x^2+y^2 \leq 1\}$ is parameterized as

$$C_1' = \{(x,y) : x(t) = \cos(t^2),\ y(t) = \sin(t^2) \text{ and } 0 \leq t \leq \sqrt{2\pi}\}$$

Solution: We see that t is not the arc length parameter since the circumference of D is 2π and yet $0 \leq t \leq \sqrt{2\pi}$. Thus, we use change of variables with

$$\int_{C_1} f\,d\omega = \int_{C_1} T\,ds = \int_{C_1'} T(x,y)\,ds$$

$$= \int_{C_1'} T(x(t), y(t))\,ds$$

$$\Rightarrow \int_{C_1} T\, ds = \int_{C_1} T(x(t), y(t)) \cdot \underbrace{\|\vec{r}\,'(t)\|_2\, dt}_{ds}$$

Side note: Calculate $\|\vec{r}\,'(t)\|_2$

$$\|\vec{r}\,'(t)\|_2^2 = (x'(t))^2 + (y'(t))^2$$

$$= (-\sin(t^2)\cdot 2t)^2 + (\cos(t^2)\cdot 2t$$

$$= 4\cdot t^2 \cdot (\sin^2(t^2) + \cos^2(t^2))$$

$$= 4 t^2$$

$$\Rightarrow \|\vec{r}\,'(t)\|_2 = \sqrt{4t^2} = 2\cdot|t| = 2t$$

since $0 \leq t \Rightarrow |t| = t$

$$\Rightarrow \int_{C_1} T\, ds = \int_{C_1} 100\cdot((x(t))^2 + 2\,(y(t))^2)\cdot 2t\, dt$$

$$= \int_0^{\sqrt{2\pi}} 100\cdot(\cos^2(t^2) + 2\sin^2(t^2))\cdot \underbrace{2t\, dt}_{du}$$

possible u-substitution

(see next page)

Let $u(t) = t^2$

$\Rightarrow \quad du = d[t^2]$

$= 2t \cdot dt$

If $t = 0 \Rightarrow u(t) = u(0) = 0$

If $t = \sqrt{2\pi} \Rightarrow u(t) = u(\sqrt{2\pi}) = 2\pi$

$$\Rightarrow \int_{C_1} T\, ds = \int_0^{2\pi} 100 \cdot (\cos^2(u) + 2\sin^2(u))\, du$$

$$= 100 \cdot \int_0^{2\pi} \cos^2(u) + 2\sin^2(u)\, du$$

$$= 100 \cdot \int_0^{2\pi} (\cos^2(u) + \sin^2(u)) + \sin^2(u)\, du$$

$$= 100 \underbrace{\int_0^{2\pi} 1 + \sin^2(u)\, du}$$

this is the same integral we worked with in Example 14.2.

Go back and redo the steps for yourself!

$$= 100 \cdot 3\pi$$

$$= 300\pi$$

"Line" integrals of scalar-valued $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}^3$

If $f : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ and $C \subseteq D$ is a curve w/ arc length parame

$$\int_C f \, ds = \int_C f(x(s), y(s), z(s)) \, ds$$

$$= \int_a^b f(x(s), y(s), z(s)) \, ds$$

However, if $C = \{ \vec{r}(t) : a \leq t \leq b \}$ and the parameter t is NOT the arc length parameter, we see we can follow similar reasoning to conclude that

$$\int_C f \, ds = \int_C f(\vec{r}(t)) \, ds \qquad \text{where} \quad ds = \|\vec{r}\,'(t)\|_2 \, dt$$

$$= \int_a^b f(\vec{r}(t)) \cdot \|\vec{r}\,'(t)\|_2 \, dt$$

$$= \int_a^b f(x(t), y(t), z(t)) \cdot \|\vec{r}\,'(t)\|_2 \, dt$$

Example 14.2.3a p. 1064 - 1065

Evaluate $\int_C (xy+2z)\,ds$ on the line segment C

from point $P(1,0,0)$ to point $Q(0,1,1)$.

Solution: Let's begin by defining our scalar-valued

$$f(x,y,z) = xy + 2z$$

We also have our curve C as a line connecting

points P to Q with

$$C = \{ \vec{l}(t) : 0 \leq t \leq 1 \}$$

$$= \{ \langle x(t), y(t), z(t) \rangle : 0 \leq t \leq 1 \}$$

Recall from Math 1C

$\vec{l}(t) = \vec{r}_0 + t \cdot \vec{v}$ where $\vec{v} = \vec{PQ}$

$$= \langle 1,0,0 \rangle + t \langle -1, 1, 1 \rangle$$

$$= \langle 1-t, t, t \rangle$$

$$\Rightarrow C = \{ \langle 1-t, t, t \rangle : 0 \leq t \leq 1 \}$$

Then, we can evaluate our line integral

$$\int_{C_1} f \, ds = \int_{C_1} f(\vec{\ell}(t)) \, ds$$

$$= \int_{C_1} f(x(t), y(t), z(t)) \, ds$$

$$= \int_{C_1} x(t) \cdot y(t) + 2z(t) \, ds$$

with $ds = \|\vec{\ell}'(t)\|_2 \, dt$

Side note: Finding $\|\vec{\ell}'(t)\|_2$

We see via our calculations

$$\vec{\ell}'(t) = \frac{d}{dt}\left[\vec{\ell}(t)\right]$$

$$= \langle -1, 1, 1 \rangle$$

$$\Rightarrow \|\vec{\ell}(t)\|_2 = \sqrt{3}$$

$$= \int_0^1 (1-t) \cdot t + 2t \cdot \sqrt{3} \, dt$$

$$= \sqrt{3} \int_0^1 t - t^2 + 2t \, dt$$

$$= \sqrt{3} \cdot \int_0^1 3t - t^2 \, dt$$

$$\Rightarrow \int_C f\,ds = \sqrt{3} \int_0^1 3t - t^2\,dt$$

$$= \sqrt{3} \left. \frac{3t^2}{2} - \frac{t^3}{3} \right|_0^1$$

$$= \sqrt{3} \left[\left(\frac{3}{2} - \frac{1}{3} \right) - \left(\frac{0}{2} - \frac{0}{3} \right) \right]$$

$$= \sqrt{3} \left(\frac{9}{6} - \frac{2}{6} \right)$$

$$= \boxed{\frac{7\sqrt{3}}{6}}$$