Lesson 11: Vector Fields

In Lesson 0 - 7, we studied integrals

$$\int_D f \, dw$$

in which $f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ focusing on multivariable, real-valued functions.

Lesson 10 was our first introduction to "integrals" of vector-valued functions:

$$\vec{F} : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}^m \quad \text{where } m = 2 \text{ or } 3$$

From here on out, we will be developing theory and techniques to integrate more general multivariable, vector-valued functions:

$$f : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

Vector Fields in $\mathbb{R}^2$

Let $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $g : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$

be multivariable, real-valued functions on a

region $D \subseteq \mathbb{R}^2$. We define a vector field

$$\vec{F} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

as a function

$$\vec{F}(x,y) = \langle f(x,y),\ g(x,y) \rangle$$

$$= f(x,y) \cdot \vec{i} + g(x,y) \cdot \vec{j}$$

Note:

- We say that vector field $\vec{F} = \langle f, g \rangle$ is continuous on $D \subseteq \mathbb{R}^2$ if BOTH f and g are continuous on $D \subseteq \mathbb{R}^2$.

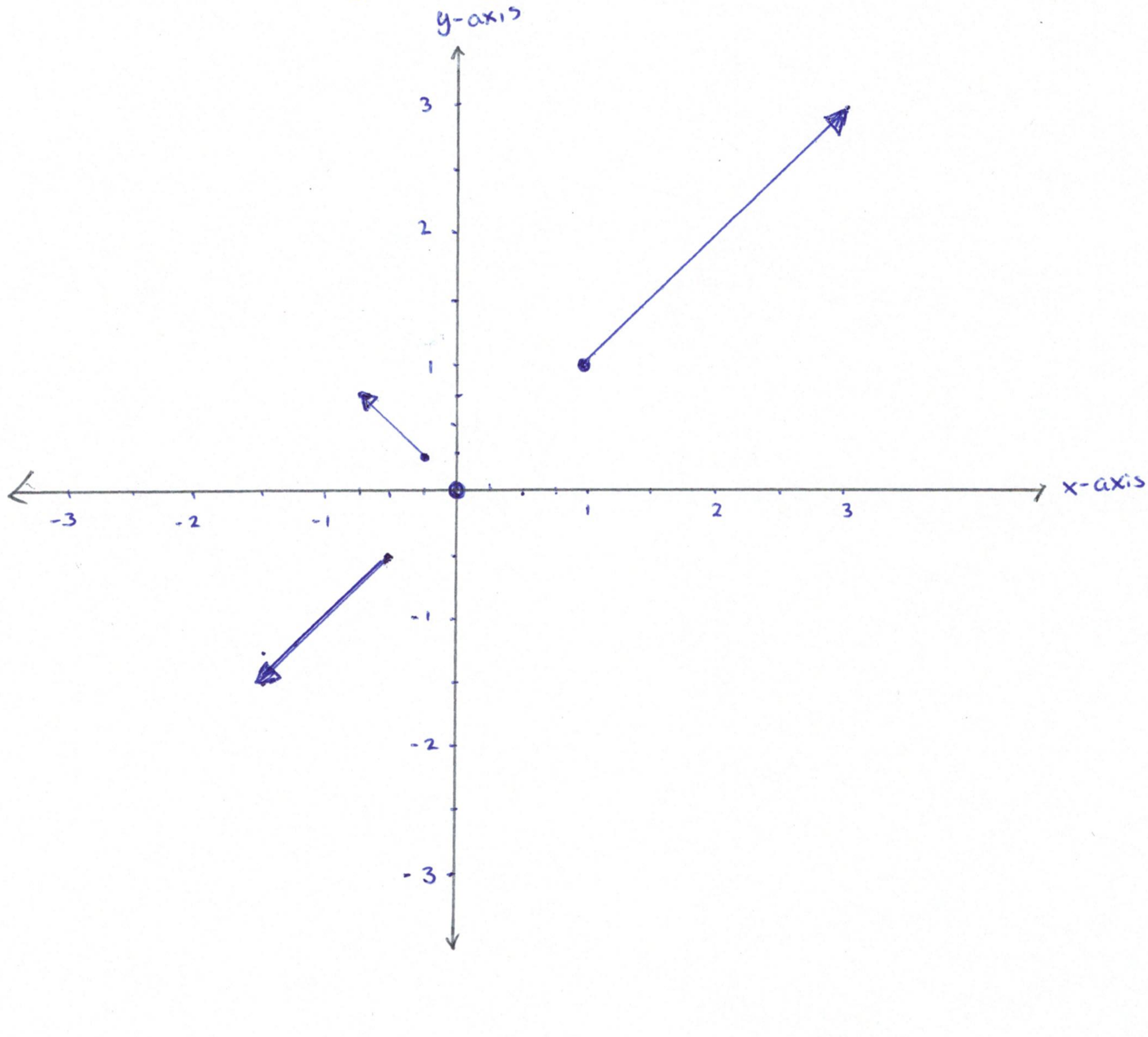
- We say that vector field $\vec{F} = \langle f, g \rangle$ is differentiable on $D \subseteq \mathbb{R}^2$ if BOTH f and g are differentiable on $D \subseteq \mathbb{R}^2$.

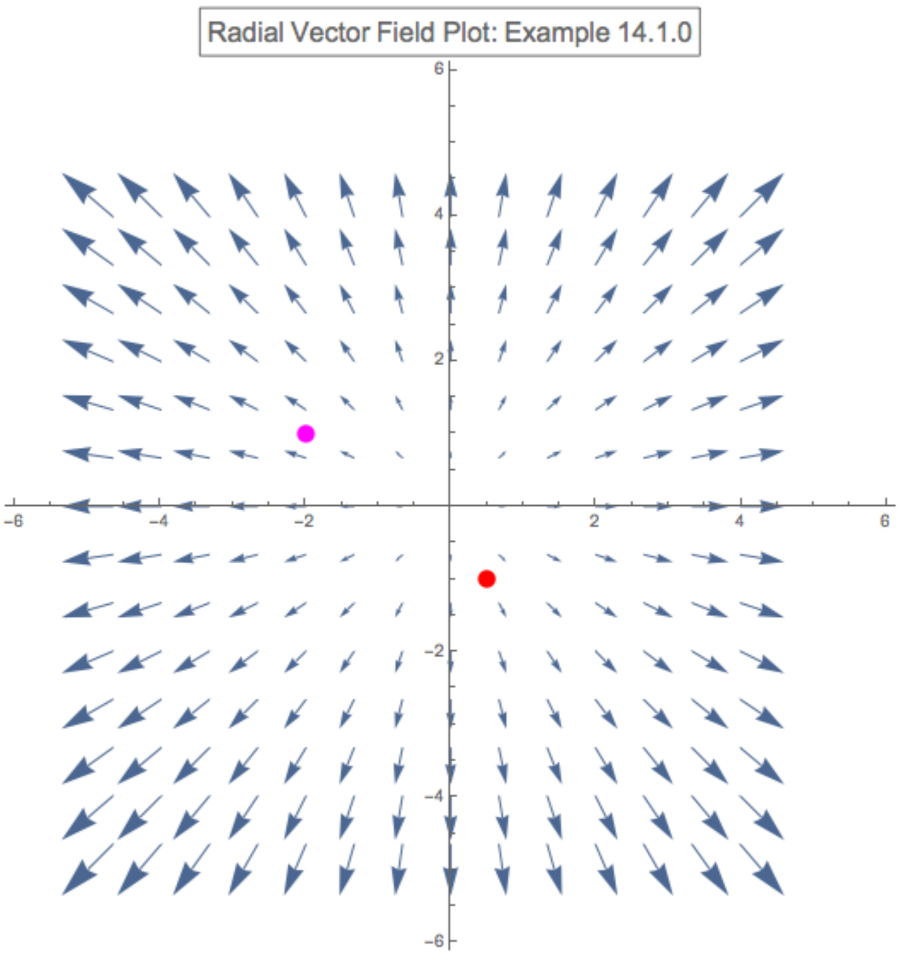
Example 14.1.0 p. 1051

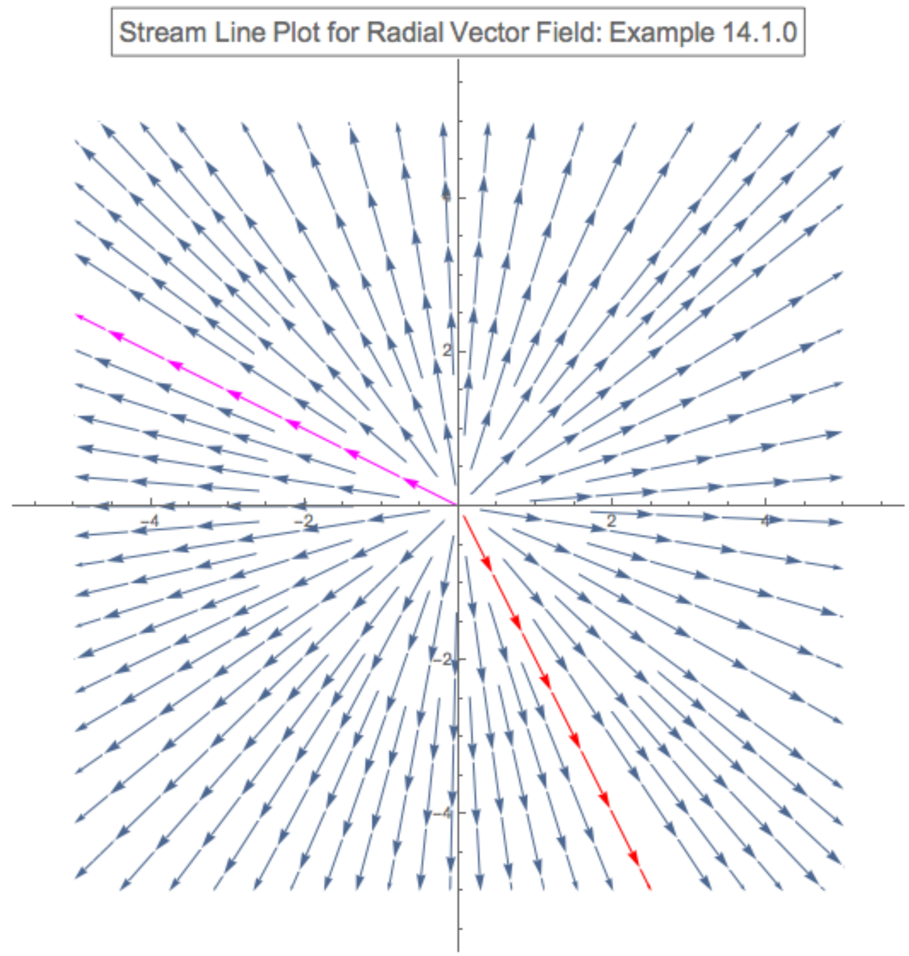
Consider the vector field

$$\vec{F}(x,y) = \langle 2x, 2y \rangle$$

Let's try to represent the "behavior" of this vector field graphically:




Radial Vector Field Plot: Example 14.1.0
6
4
2
-2
-4
-6
-6
-4
-2
2
4
6


Stream Line Plot for Radial Vector Field: Example 14.1.0
4
2
-2
-4
-4
-2
2
4

At a selected input point $P(x,y)$, we plot the

output vector $\vec{F}(x,y)$ with a tail at $P(x,y)$

For example, at input $P(1,1)$, we see

$$\vec{F}(1,1) = \langle 2\cdot 1,\ 2\cdot 1 \rangle = \langle 2,2 \rangle$$

Then, we draw a vector with tail at $P(1,1)$

and head at $\langle 1,1 \rangle + \langle 2,2 \rangle = \langle 3,3 \rangle$

Input value (x,y)	output $\vec{F}(x,y) = \langle 2x, 2y \rangle$
$(x,y) = (0,0)$	$\vec{F}(0,0) = \langle 0, 0 \rangle$
$(x,y) = (1,1)$	$\vec{F}(1,1) = \langle 2, 2 \rangle$
$(x,y) = \left(\frac{-1}{4}, \frac{1}{4}\right)$	$\vec{F}\left(\frac{-1}{4}, \frac{1}{4}\right) = \left\langle -\frac{2}{4}, \frac{2}{4} \right\rangle = \left\langle -\frac{1}{2}, \frac{1}{2} \right\rangle$
$(x,y) = \left(-\frac{1}{2}, -\frac{1}{2}\right)$	$\vec{F}\left(-\frac{1}{2}, -\frac{1}{2}\right) = \left\langle -\frac{2}{2}, -\frac{2}{2} \right\rangle = \langle -1, -1 \rangle$

Notice, this form of graphing is very different then the work we did with $f: D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$

Notice, for the vector field $\vec{F}(x,y) = \langle 2x, 2y \rangle$,

- For every nonzero $(x,y) \in \mathbb{R}^2$, the vector $\vec{F}(x,y)$ points in the direction $\langle 2x, 2y \rangle$ which is directly away from the origin.

- The length of $\vec{F}(x,y)$ at $P(x,y)$ is

$$\|\vec{F}(x,y)\|_2 = \|\langle 2x, 2y \rangle\|_2$$

$$= \sqrt{(2x)^2 + (2y)^2}$$

$$= \sqrt{4x^2 + 4y^2}$$

$$= \sqrt{4 \cdot (x^2 + y^2)}$$

$$= \sqrt{4} \cdot \sqrt{x^2 + y^2} = 2 \cdot \sqrt{x^2 + y^2}$$

Definition Radial Vector Fields in $\mathbb{R}^2$

Let $\vec{r}(x,y) = \langle x, y \rangle$ where $\vec{r} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Let $f(x,y)$ be a two-variable, real-valued function

where $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$.

A vector field in the form

$$\vec{F}(x,y) = f(x,y) \cdot \vec{r}$$

$$= f(x,y) \cdot \langle x, y \rangle$$

is called a radial vector field.

Remark: Of special interest are the radial vector fields

$$\vec{F}(x,y) = \frac{\vec{r}}{\|\vec{r}\|_2^p} = \frac{\langle x, y \rangle}{\|\vec{r}\|_2^p}$$

where $p \in \mathbb{R}$. At every point $(x,y) \in \mathbb{R}^2$, the vectors are pointed directly outward from origin w/ $\|\vec{F}\| = \dfrac{1}{\|\vec{r}\|_2^{p-1}}$

see Mathematica notebook for clear code to produce visuals

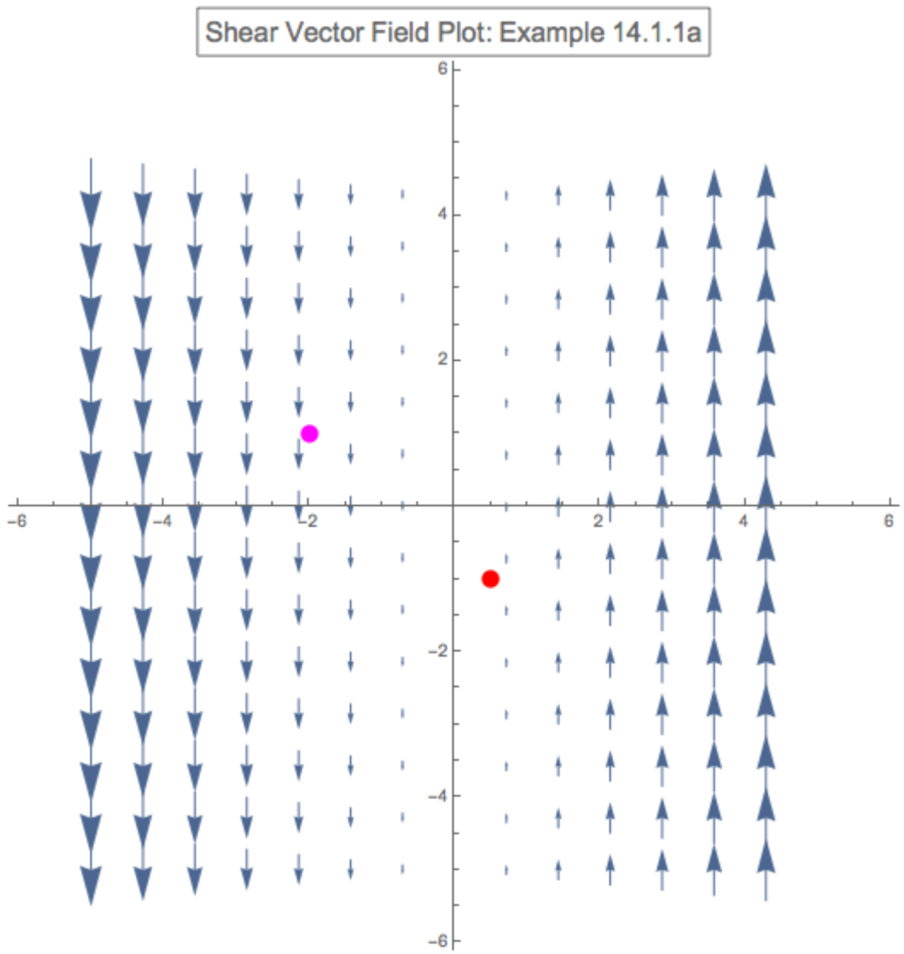
Example 14.1.1 p. 1052

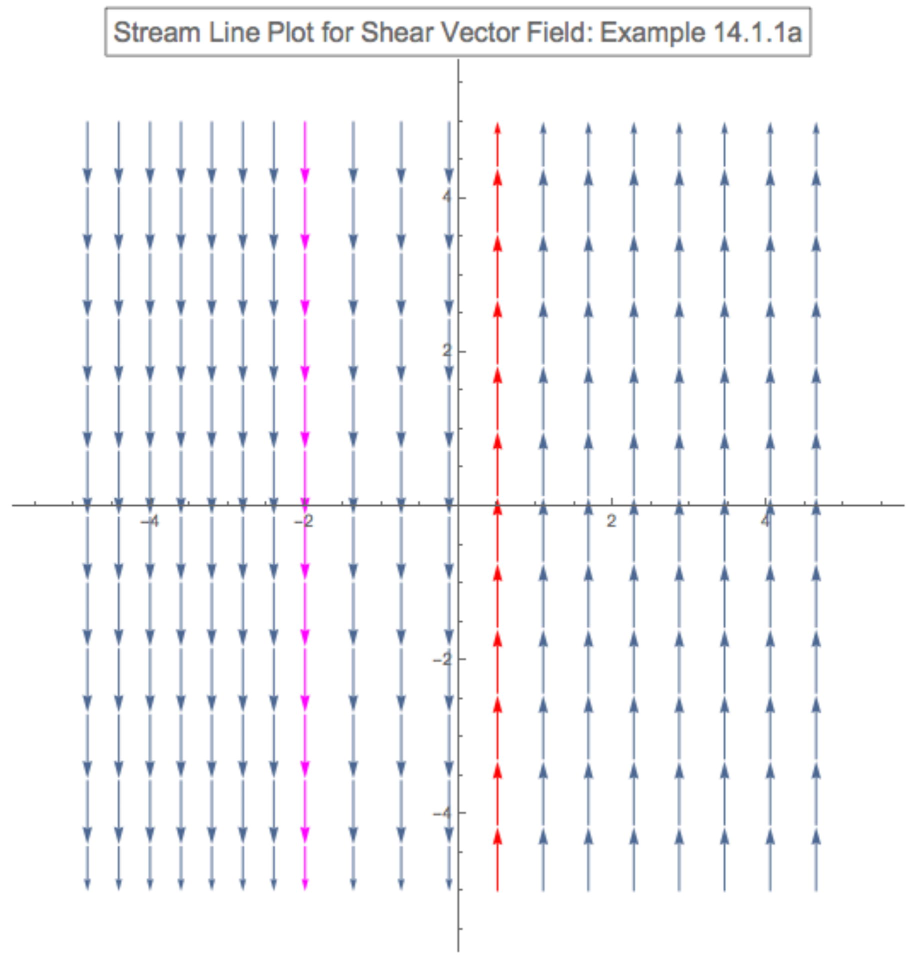
Sketch "representative" vectors from each of the following vector fields:

A. $\vec{F}(x,y) = \langle 0, x \rangle$ ← shear field

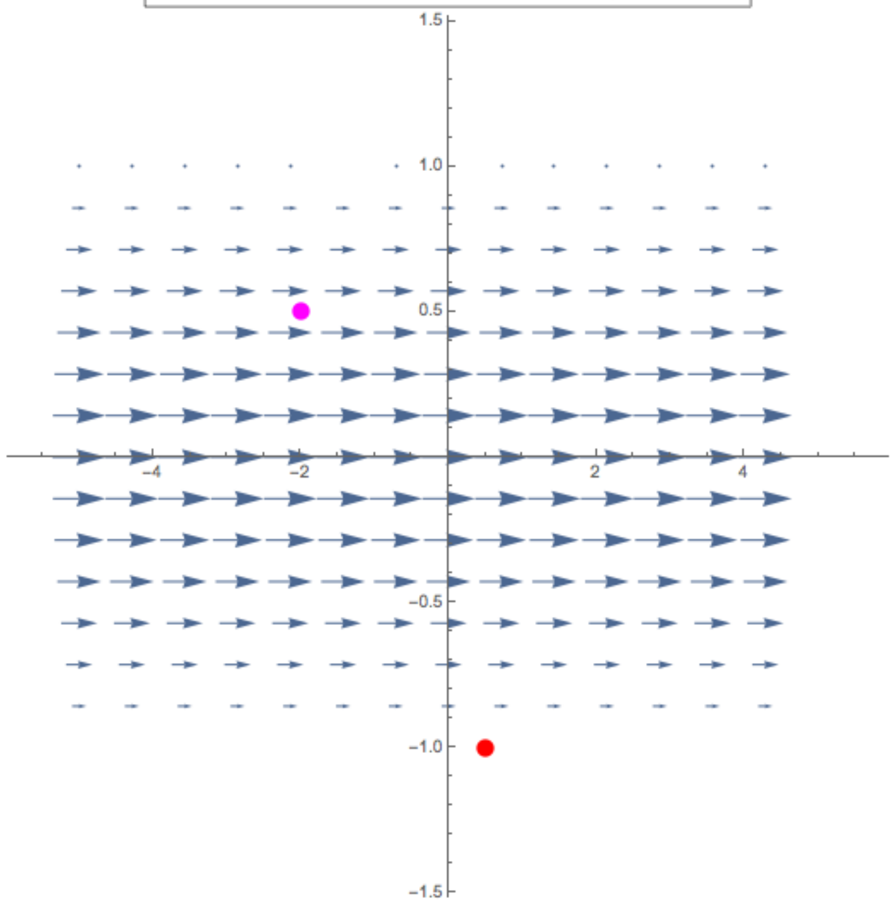
channel flow → B. $\vec{F}(x,y) = \langle 1-y^2, 0 \rangle$, $|y| \leq 1$

C. $\vec{F}(x,y) = \langle -y, x \rangle$ ← rotational field

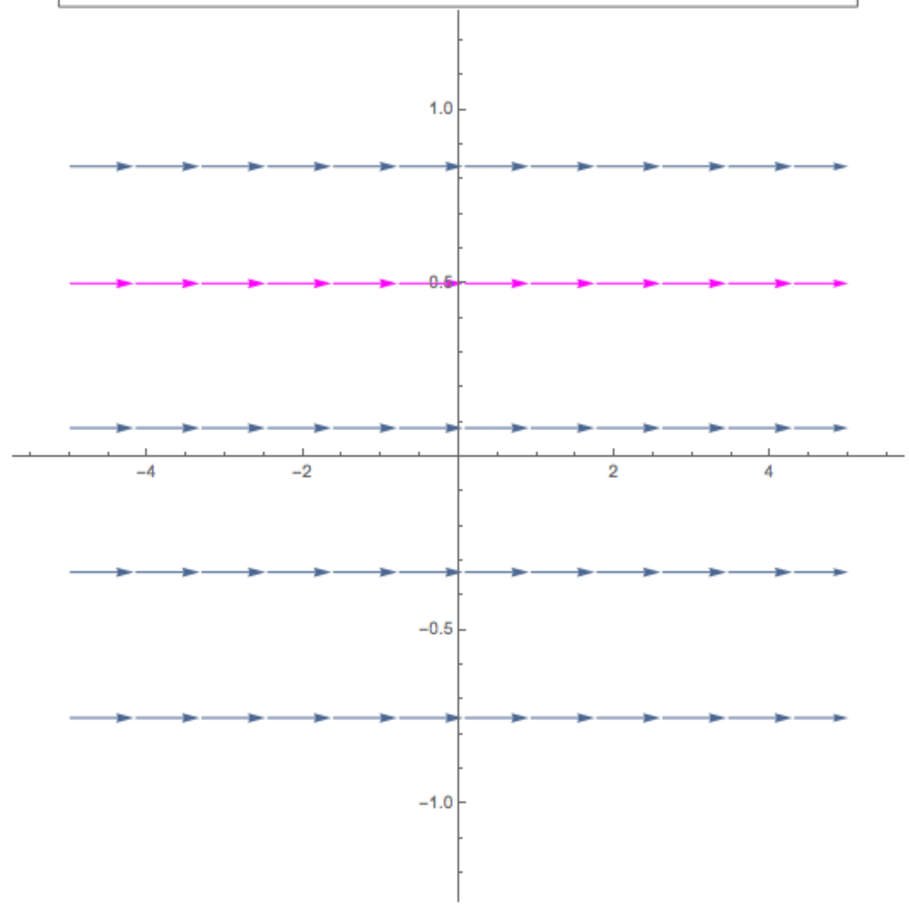

Shear Vector Field Plot: Example 14.1.1a
6
4
2
−2
−4
−6
−6
−4
−2
2
4
6


Stream Line Plot for Shear Vector Field: Example 14.1.1a
4
2
−2
−4
−4
−2
2
4

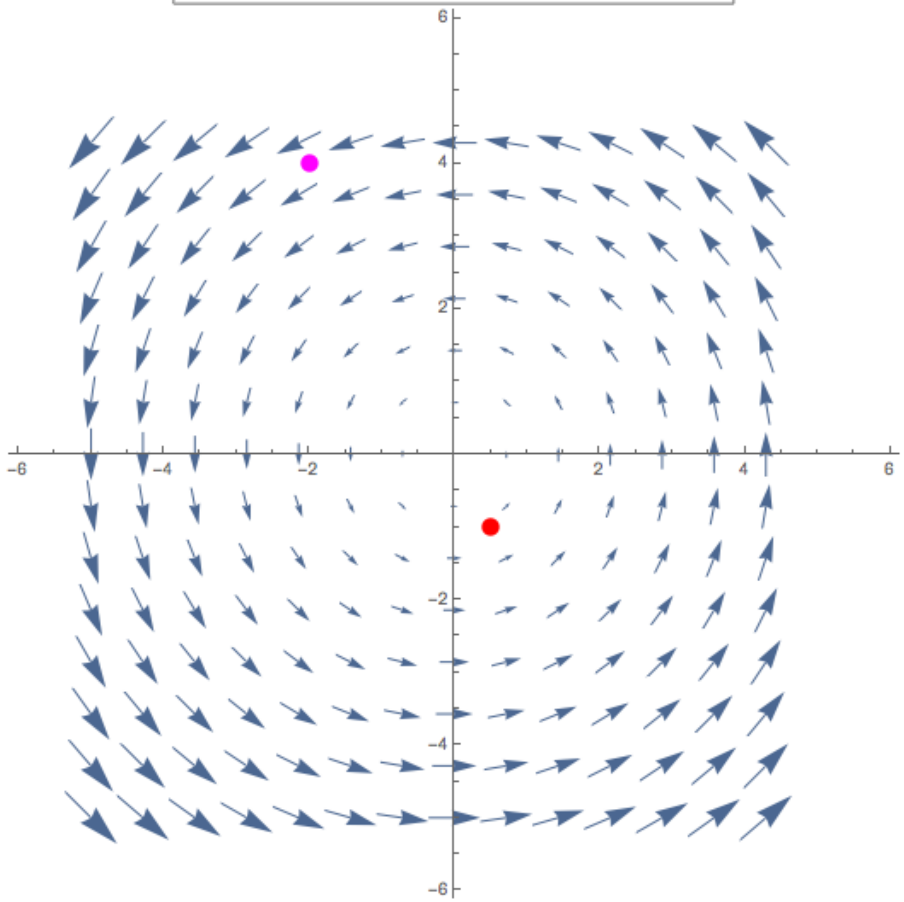
Channel Flow Vector Field Plot: Example 14.1.1b



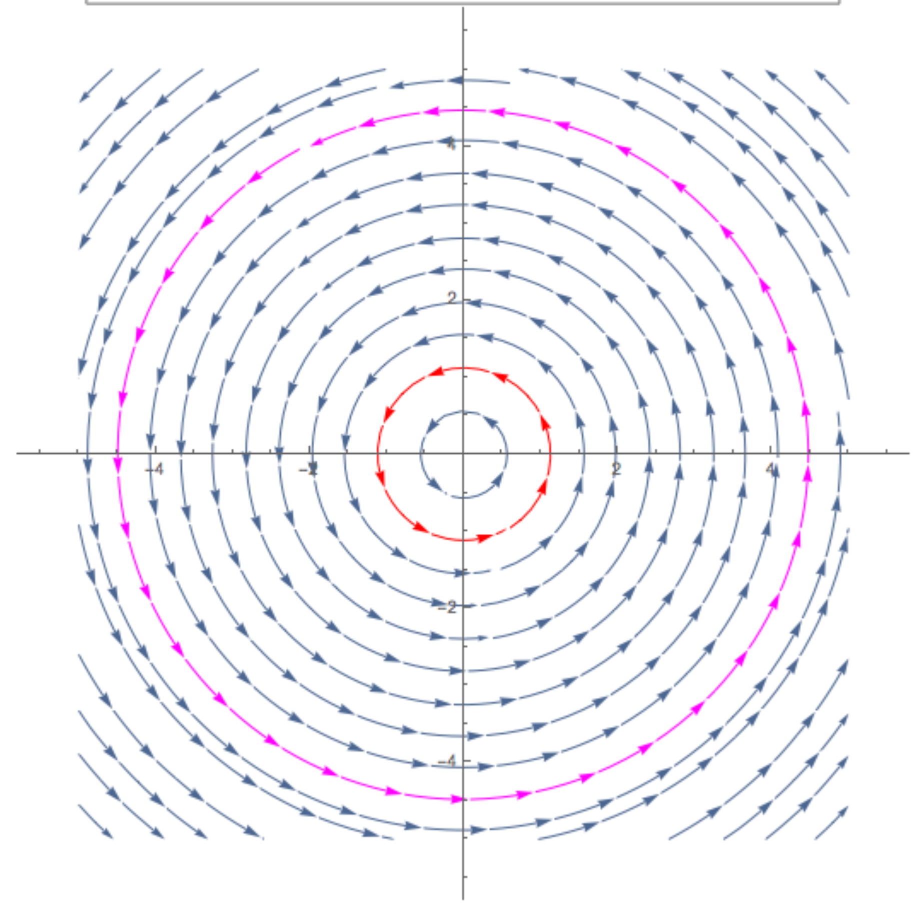
Stream Line Plot for Channel Flow Vector Field: Example 14.1.1b



Rotational Vector Field Plot: Example 14.1.1c



Stream Line Plot for Rotational Vector Field: Example 14.1.1c



Gradient Fields and Potential Functions

Let $\varphi : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a two-variable, real-valued function. Suppose we visualize the output of this function as a surface

$$z = \varphi(x,y)$$

Recall that we can visualize the behavior of the surface by graphing various level curves:

$$L_c(\varphi) = \{ (x,y) \in D : \varphi(x,y) = c \text{ for some } c \in \mathbb{R} \}$$

At the point (a,b) on a specific level curve, the gradient $\vec{\nabla}\varphi(a,b) = \langle \varphi_x(a,b), \varphi_y(a,b) \rangle$ is orthogonal to the (tangent line of the) level curve at the point (a,b).

With this geometry in mind, one way to "generate" vector fields is to let

$$\vec{F}(x,y) = \vec{\nabla}\varphi(x,y)$$

$$= \langle \varphi_x(x,y), \varphi_y(x,y) \rangle$$

$$= \langle f(x,y), g(x,y) \rangle$$

- Such a vector field $\vec{F} = \vec{\nabla}\varphi$ is called a gradient field (since the field arises from taking the gradient of some scalar function)

- The scalar function $\varphi = \varphi(x,y)$ is called a potential function.

Gradient fields are useful in many applications because of the physical meaning of the gradient.

Example: Suppose $\varphi = \varphi(x,y)$ represents the temperature of a "point" (x,y) in the cross section of a conducting material.

The gradient field $\vec{F} = \vec{F}(x,y) = \vec{\nabla}\varphi(x,y)$ evaluated at the point (x,y) gives the "direction" (within the domain $D \subseteq \mathbb{R}^2$ of φ) in which the temperature increases most rapidly at that point.

There is a "basic" law of physics which states that heat diffuses in the direction of the vector field

$$-\vec{F} = -\vec{F}(x,y) = -\vec{\nabla}\varphi(x,y)$$

the direction in which the temperature decreases most rapidly.

☐ Watch youtube video before talking about this example

Example 14.1.4 a p. 1056

Suppose $T(x,y) = 200 - x^2 - y^2$ where $T: D \subseteq \mathbb{R}^2 \to \mathbb{R}$

and $D = \{(x,y) : x^2 + y^2 = 25\}$

Sketch and interpret the gradient field.

Solutions: