Name : ____________________ **Class Number**: _____

Math 1D: Lesson 10 Suggested Problems

Theoretic Problems: Discussed in-class

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1. Suppose that we have a parameterized curve

$$C = \{\, \mathbf{r}(t) \,:\, a \le t \le b \,\}$$

where $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. Suppose that the functions $x(t)$ and $y(t)$ have continuous derivatives for all $t \in [a, b]$. Suppose L is the total arc length of the curve C. Explain how to find L using a definite integral. In particular, answer each of the following questions:

   i. Explain how to set up a partition of the interval $[a, b]$ into n subregions with

   $$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

   ii. Draw a diagram of a general curve $C \subseteq \mathbb{R}^2$. Label point P_k with coordinates $\big(x(t_k), y(t_k)\big)$ for $k = 1, 2, 3, ..., n$. Let L_k be the exact length of the kth sub-arc connecting point $P_{k-1}\big(x(t_{k-1}), y(t_{k-1})\big)$ to point $P_k\big(x(t_k), y(t_k)\big)$ on the curve C.

   iii. Explain how to construct the following approximation

   $$L_k \approx \sqrt[2]{\ (\Delta x_k)^2 + (\Delta y_k)^2}$$

   where $\Delta x_k = x(t_k) - x(t_{k-1})$ and $\Delta y_k = y(t_k) - y(t_{k-1})$.

   iv. Using steps i. - iii. above, explain why we can approximate L, the arc length of the curve C, as

   $$L \approx \sum_{k=1}^{n} \sqrt[2]{\ (\Delta x_k)^2 + (\Delta y_k)^2}$$

   v. Recall the precise statement of the Mean Value Theorem from Math 1A. Specifically, go back into your math 1A text book and rewrite the mean value theorem word-for-word in your notes. Then, translate the formal theorem statement into your own words.

   vi. Suppose $\Delta t_k = t_k - t_{k-1}$. Using the mean value theorem from Math 1A, explain why we know there exists a point $t_k^\star \in (t_{k-1}, t_k)$ such that

   $$\Delta x_k = x(t_k) - x(t_{k-1}) = x'(t_k^\star)\,\Delta t_k$$

   vii. Explain how we know there must be a second, possibly different, $\widehat{t_k} \in (t_{k-1}, t_k)$ such that

   $$\Delta y_k = y(t_k) - y(t_{k-1}) = y'(\widehat{t_k})\,\Delta t_k$$

   viii. Using steps v. - vii. above, explain why we can approximate L, the arc length of the curve C, as

   $$L \approx \sum_{k=1}^{n} \sqrt[2]{\ (\Delta x'(t_k^\star))^2 + \big(y'(\widehat{t_k})\big)^2}\ \ \Delta t_k$$

   Make sure to show all algebraic steps needed to bring the factor Δt_k out of the radicand.

   ix. Explain how to use a limit to derive the precise arc length formula

   $$L = \int_a^b \|\mathbf{r}'(t)\|_2 \ dt$$

   Make sure to explain how the limit affects each term in the approximation and results in the stated definition for calculating the arc length L of the curve C.

2. Suppose that we have a parameterized curve

$$C = \{\, \mathbf{r}(t) \,:\, a \leq t \leq b \,\}$$

where $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. Suppose that the functions $x(t)$ and $y(t)$ have continuous derivatives for all $t \in [a, b]$. Answer each of the following questions:

   i. Describe how and why the the arc length function

   $$s(t) = \int_a^t \|\mathbf{r}'(u)\|_2 \; du$$

   measures the distance between point $\mathbf{r}(a)$ and point $\mathbf{r}(t)$ for any $a \leq t \leq b$.

   ii. Recall the precise statement of both parts of the Fundamental Theorem of Calculus from Math 1A. Specifically, go back into your math 1A text book and rewrite both parts of the fundamental theorem of calculus word-for-word in your notes. Then, translate the formal theorem statement into your own words.

   iii. Using the fundamental theorem of calculus, explain why we know that

   $$s'(t) = \|\mathbf{r}'(t)\|_2$$

   iv. Use parts i. - iii. above to describe when can we conclude that the parameter t corresponds to the arc length of the curve C.

Problems Solved in Jeff's Handwritten Notes

3. Example 11.8.1 p.832: Find circumference of a circle with radius a using the arc length formula
4. Example 11.8.5$\frac{1}{2}$ p.837:
5. Example 11.8.6 p. 838

Suggested Problems: Answers in Book

6. Example 11.8.2 p. 833
7. Example 11.8.4 p. 834 - 835
8. Exercise 11.8.2 p. 838
9. Exercise 11.8.7 p. 838
10. Exercise 11.8.43 p. 839
11. Exercise 11.8.47 p. 839

Optional Challenge Problems

12. Exercise 11.8.62 p. 840
13. Exercise 11.8.65 p. 840
14. Exercise 11.8.66 p. 840
15. Prove the mean value theorem
16. Prove the fundamental theorem of calculus.