Lesson 10 : Lengths of (parametrized) curves

Suppose we have a parametrized curve $C_1 \subseteq \mathbb{R}^2$

defined by a single-variable, vector valued function

$$\vec{r} : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$$

where $\vec{r}(t) = \langle x(t), y(t) \rangle$ and $t \in D = [a,b]$

with $a \leq t \leq b$.

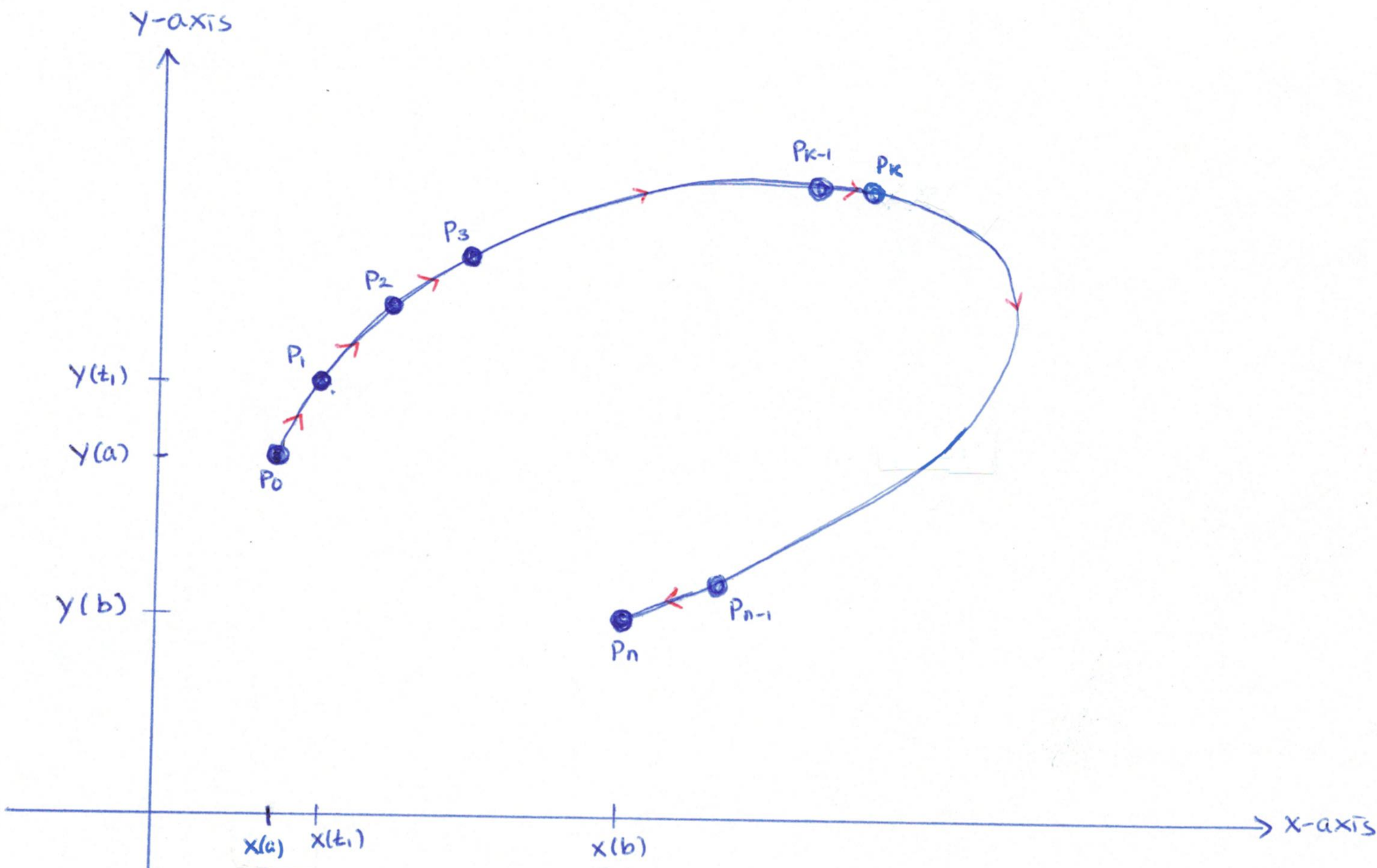
Suppose the derivatives $x'(t) = \frac{d}{dt}[x(t)]$ and $y'(t) = \frac{d}{dt}[y(t)]$

are continuous on interval $[a,b]$.

Let's find the arc length of C given by

$\vec{r}(t) = \langle x(t), y(t) \rangle$ for all $a \le t \le b$.

We begin with a general geometric interpretation of this problem:



The total arc length L of curve $C = \{ \vec{r}(t) : a \le t \le b \}$ is the sum of the arc lengths of each subarc.

To find the length of the length of the curve C between the points $P_a(x(a), y(a))$ and $P_b(x(b), y(b))$, we will begin by subdividing the interval $[a, b]$ into n subintervals using the "grid points"

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

Next, we will sample our vector-valued function $\vec{r}(t)$ at each of these grid points to create point P_K on curve

Starting point | Point 0 : $P_0(x(t_0), y(t_0)) = \vec{r}(t_0)$

Point 1 : $P_1(x(t_1), y(t_1)) = \vec{r}(t_1)$

orientation | Point 2 : $P_2(x(t_2), y(t_2)) = \vec{r}(t_2)$

$\vdots$

Point K : $P_K(x(t_K), y(t_K)) = \vec{r}(t_K)$

$\vdots$

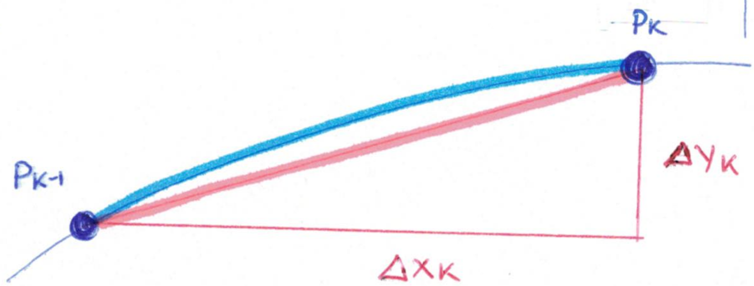
End point | Point n : $P_n(x(t_n), y(t_n)) = \vec{r}(t_n)$

Consider the following problem:

Approximate the arc length of a "slightly curved" arc between two known points

Point	coordinates
P_K	$= (x(t_K), y(t_K))$
P_{K-1}	$= (x(t_{K-1}), y(t_{K-1}))$

length of blue arc is desired



Let L_K be length of blue arc

$\Rightarrow$ $L_K \approx$ length of red line segment

$$\Rightarrow \quad L_K \approx \sqrt{\Delta x_K^2 + \Delta y_K^2}$$

Notice that

$$\Delta x_K = x(t_K) - x(t_{K-1})$$

$$\Delta y_K = y(t_K) - y(t_{K-1})$$

By the mean value theorem from math 1A, we know that on the interval $[t_{k-1}, t_k]$, there is some interior point $t_k^* \in (t_{k-1}, t_k)$ such that

$$x'(t_k^*) = \frac{x(t_k) - x(t_{k-1})}{t_k - t_{k-1}} = \frac{\Delta x_k}{\Delta t_k}$$

$$\Rightarrow \quad \Delta x_k = x'(t_k^*) \cdot \Delta t_k = x(t_k) - x(t_{k-1})$$

Similary, we know there is some other $\hat{t}_k \in (t_{k-1}, t_k)$ such that

$$y'(\hat{t}_k) = \frac{y(t_k) - y(t_{k-1})}{t_k - t_{k-1}} = \frac{\Delta y_k}{\Delta t_k}$$

$$\Rightarrow \quad \Delta y_k = y'(\hat{t}_k) \cdot \Delta t_k$$

Then the total length of the curve C will be given by

$$L = \sum_{k=1}^{n} L_k$$

$$\approx \sum_{k=1}^{n} \sqrt{\Delta x_k^2 + \Delta y_k^2}$$

$$= \sum_{k=1}^{n} \sqrt{\left(x'(t_k^*) \cdot \Delta t_k\right)^2 + \left(y'(\hat{t}_k) \cdot \Delta t_k\right)^2}$$

$$= \sum_{k=1}^{n} \sqrt{(x'(t_k^*))^2 + (y'(\hat{t}_k))^2} \cdot \Delta t_k$$

Note:
since $t_{k-1} < t_k$
$\Rightarrow 0 < t_k - t_{k-1}$
$\Rightarrow 0 < \Delta t_k$
$\Rightarrow |\Delta t_k| = \Delta t_k$
$\Rightarrow \sqrt{\Delta t_k^2} = \Delta t_k$

Then, if we set $\Delta = \max\{\Delta t_1, \Delta t_2, \cdots, \Delta t_n\}$, we can find the exact curve length by taking the limit:

$$L = \lim_{\Delta \to 0} \sum_{k=1}^{n} \sqrt{(x'(t_k^*))^2 + (y'(\hat{t}_k))^2} \cdot \Delta t_k$$

Recall that the interior points $t_k^*, \hat{t}_k \in (t_{k-1}, t_k)$

came from the mean value theorem and eventually

$$t_k^* \to t \qquad \text{and} \qquad \hat{t}_k \to t$$

for some t in the original interval since $\Delta t_k \to 0$ as $\Delta \to 0$.

Thus, we see we can measure the exact arc length

via an Integral

$$L = \lim_{\Delta \to 0} \sum_{k=1}^{n} \sqrt{(x'(t_k^*))^2 + (y'(\hat{t}_k))^2} \cdot \Delta t_k$$

$$= \int_a^b \sqrt{x'(t) + y'(t)}\, dt$$

$$= \int_a^b \| \vec{r}\,'(t) \|_2 \, dt$$

$$= \int_a^b \| \vec{v}\,'(t) \|_2 \, dt \qquad \text{where } \vec{v}(t) = \frac{d}{dt}\left[\vec{r}(t)\right]$$

Example 11.8.1 p. 832)

Prove the circumfrence of a circle of radius $a>0$ is $2\pi a$

Solution: Recall that the vector-valued equation for a circle is

$$\vec{r}(t) = \langle a \cdot \cos(t), a \cdot \sin(t) \rangle = \langle x(t), y(t) \rangle$$

for $0 \leq t \leq 2\pi$.

We know then that $\begin{cases} x'(t) = -a\sin(t) \\ y'(t) = a\cos(t) \end{cases}$

$$\Rightarrow L = \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(-a\sin(t))^2 + (a\cos(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2 \sin^2(t) + a^2 \cos^2(t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2 \cdot (\cos^2(t) + \sin^2(t))}\, dt = \int_0^{2\pi} a\, dt = \boxed{2\pi a} \checkmark$$

Arc Length as a Parameter p. 836 - 838

Recall that if $\vec{r} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a vector valued function with

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

we derived a formula for the arc length of curve

$$C = \{ \langle x(t), y(t) \rangle : a \leq t \leq b \}.$$

We saw that if arc length$(C) = L$, then

$$L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$$

$$= \int_a^b \| \vec{r}\,'(t) \|_2 \, dt$$

$$= \int_a^b \| \vec{v}(t) \|_2 \, dt$$

In this vector-valued function, the parameter $t \in D = [a,b] \subseteq \mathbb{R}$ was chosen as a convienent variable and suggested a connection to time.

However, there are some annoying aspects about the choices of t as the parameter. To better understand the limitations of t as a parameter, Let's take a look at some challenges posed by the choice of variable t

Example 13.7.5½ p. 837

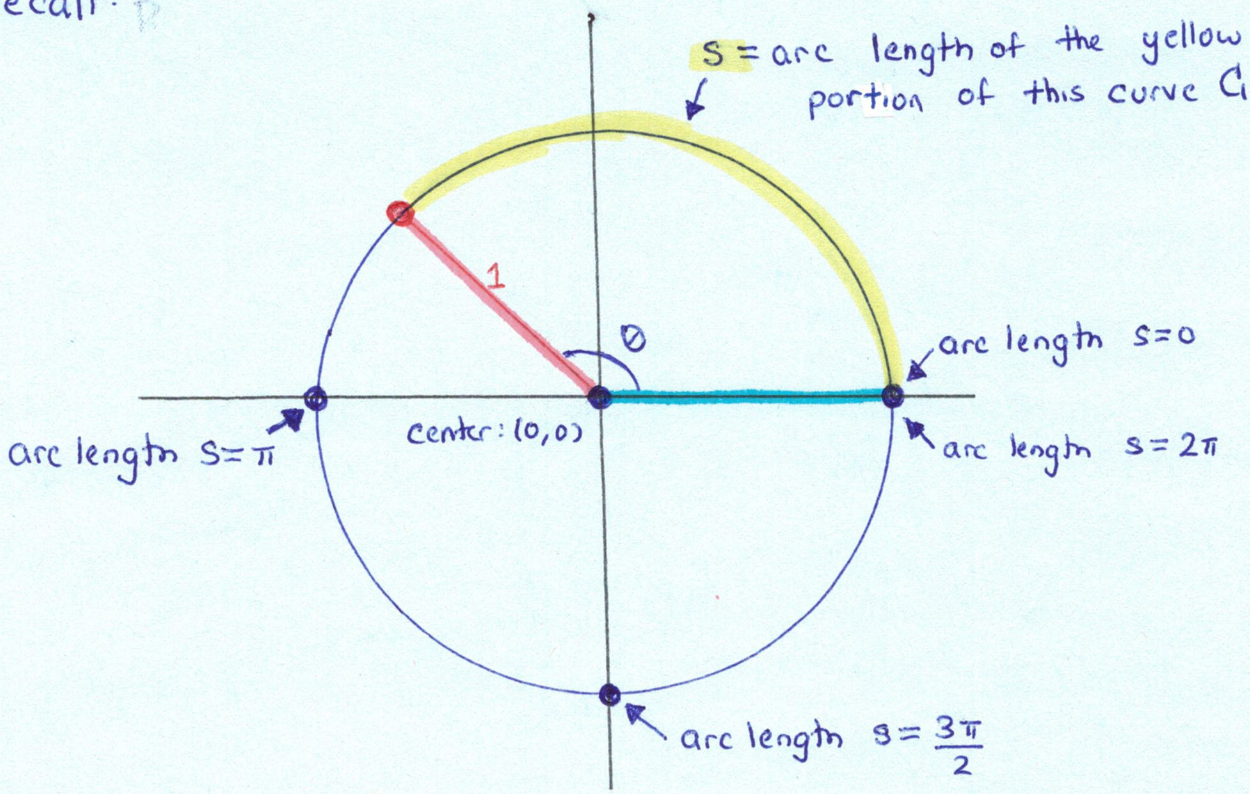
Consider the unit circle with radius 1 centered at the origin. Suppose we have three "parameterizations" of this curve given by

$$\vec{r}_1(t) = \langle \cos(t), \sin(t) \rangle \quad \text{with} \quad 0 \leq t \leq 2\pi$$

$$\vec{r}_2(t) = \langle \cos(2t), \sin(2t) \rangle \quad \text{with} \quad 0 \leq t \leq \pi$$

$$\vec{r}_3(t) = \langle \cos\left(\tfrac{t}{2}\right), \sin\left(\tfrac{t}{2}\right) \rangle \quad \text{with} \quad 0 \leq t \leq 4\pi$$

Recall:

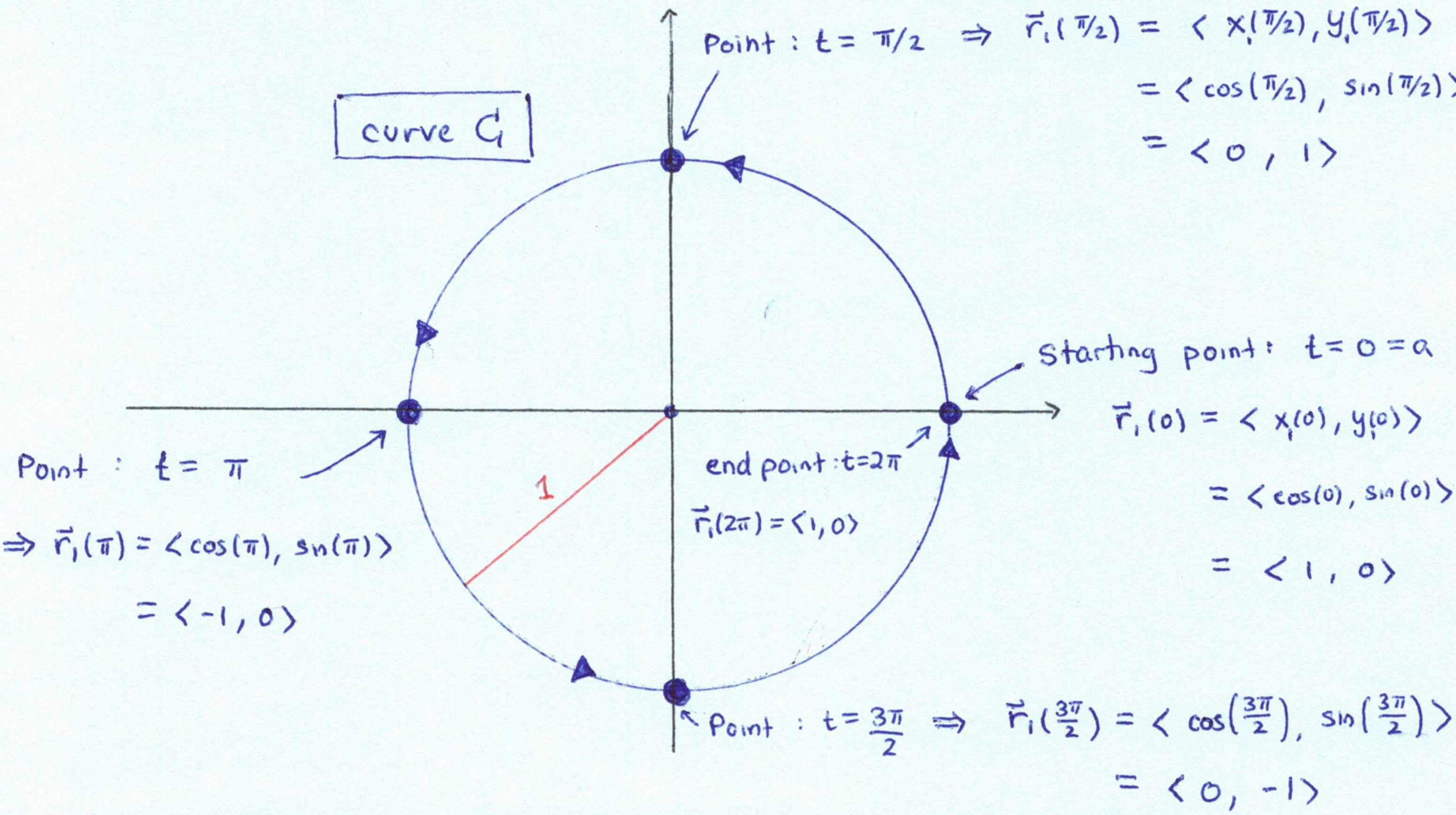


Let's introduce a new parameter s to represent the "arc length" of the curve highlighted in yellow.

In our past lives, we might have thought of s as "dependent" on parameter θ: $s = s(\theta)$.

However, we can think of s as an independent variable, the "distance" traveled along a curve that does not necessarily depend on θ.

Let's take a look at each of these curves:

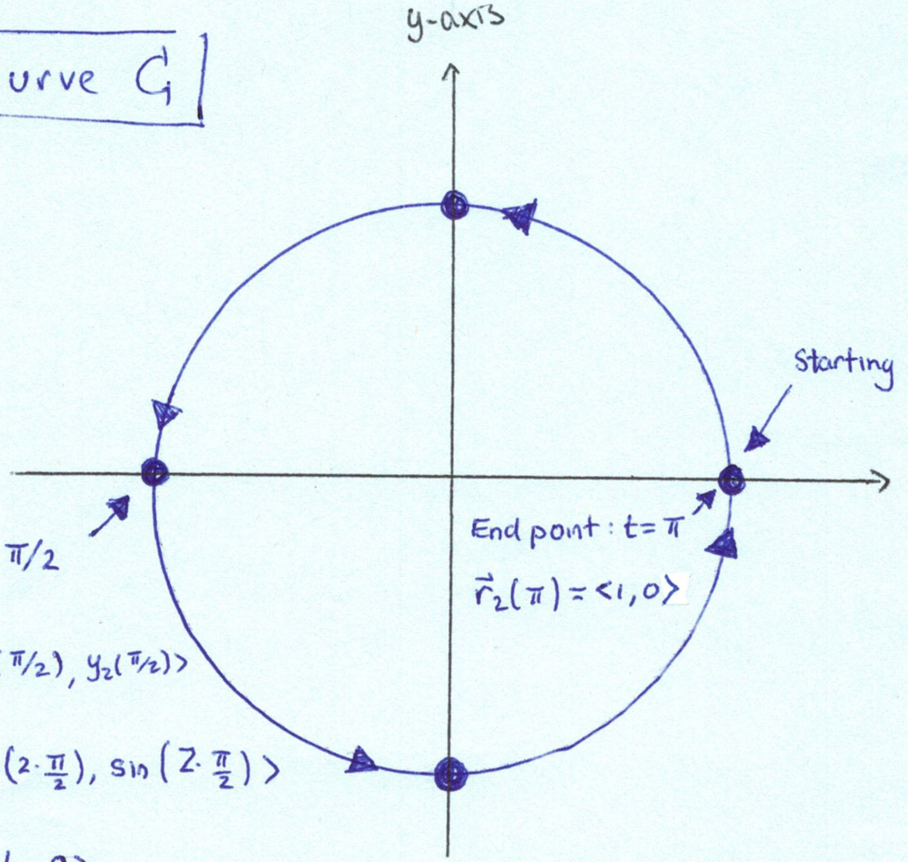


Curve $C_1 = \{ \vec{r}_1(t) : 0 \leq t \leq 2\pi \}$

- ☐ The curve C_1 is the (oriented) circle where orientation is implicit in $\vec{r}_1(t)$
- ☐ The arc length of the curve C_1 is the circumference of this circle which is 2π

- ☐ The arc length $s = 0$ corresponds to parameter $t = 0$
- ☐ The arc length $s = 2\pi$ corresponds to parameter value $t = 2\pi$.

Curve C_1



$\vec{r}_2(0) = < x_2(0), y_2(0)>$

$= <\cos(2\cdot 0), \sin(2 \cdot 0)>$

$= <1, 0>$

Point : $t = \pi/2$

$\vec{r}_2(\pi/2) = < x_2(\pi/2), y_2(\pi/2)>$

$= <\cos(2\cdot \frac{\pi}{2}), \sin(2\cdot \frac{\pi}{2})>$

$= <-1, 0>$

$$\text{Curve } C_1 = \{ \vec{r}_2(t) : 0 \le t \le \pi \}$$

- ☐ The curve C_1 is the same "oriented circle" where the function $\vec{r}_2(t)$ defines positive orientation "in the same direction" as the orientation for $\vec{r}_1(t)$.

- ☐ The arclength of C_1 as described by $\vec{r}_2(t)$ is the same as the arclength of C_1 as described by $\vec{r}_1(t)$ which is the circumference of the circle : $2\pi = S$

- ☐ Arc length $s = 0$ corresponds to parameter $t = 0$
- ☐ Arc length $s = 2\pi$ corresponds to parameter $t = \pi$.

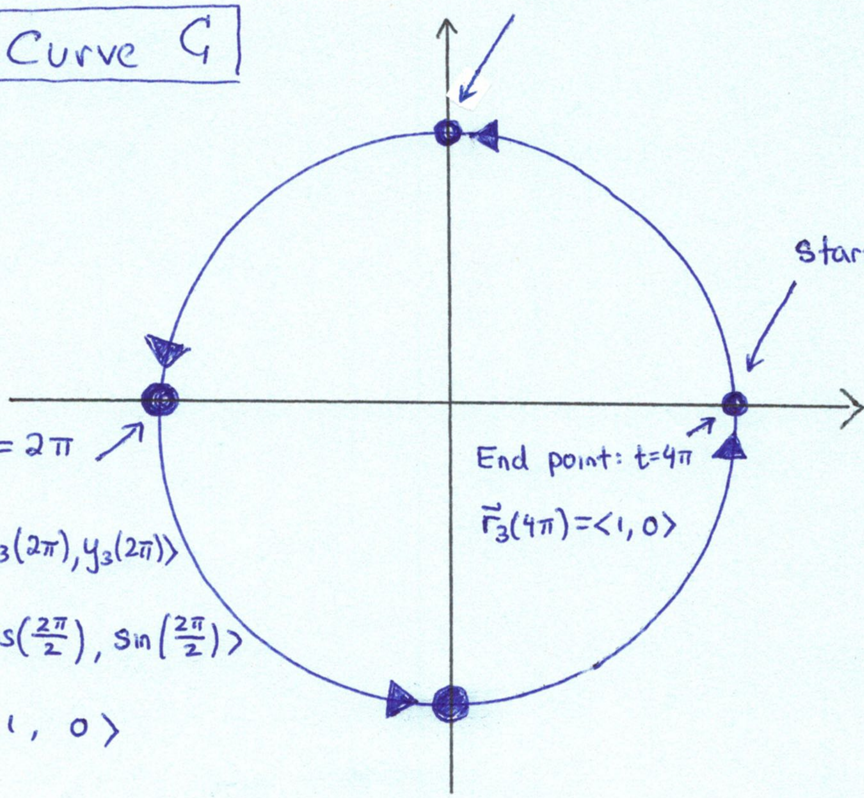
notice the difference here!

Curve C_1

Point: $t = \pi \Rightarrow \vec{r}_3(\pi) = \langle x_3(\pi), y_3(\pi) \rangle$

$= \langle \cos(\frac{\pi}{2}), \sin(\frac{\pi}{2}) \rangle$

$= \langle 0, 1 \rangle$



Starting point : $t = 0$

$\vec{r}_3(0) = \langle x_3(0), y_3(0) \rangle$

$= \langle \cos(\frac{0}{2}), \sin(\frac{0}{2}$

$= \langle 1, 0 \rangle$

point : $t = 2\pi$

$\vec{r}_3(2\pi) = \langle x_3(2\pi), y_3(2\pi) \rangle$

$= \langle \cos(\frac{2\pi}{2}), \sin(\frac{2\pi}{2}) \rangle$

$= \langle -1, 0 \rangle$

$$\text{Curve } C_1 = \{ \vec{r}_3(t) : 0 \le t \le 4\pi \}$$

- The curve C_1 is the same exact "oriented circle" with identical start and end points and same positive orientation when encoded using function $\vec{r}_3(t)$ as with functions $\vec{r}_1(t)$ or $\vec{r}_2(t)$
- The arclength of C_1 as described by $\vec{r}_3(t)$ is the same as before, which is 2π.
- Arclength $s=0$ corresponds to parameter $t=0$ in $\vec{r}_3(t)$.
- Arclength $s = 2\pi$ corresponds to parameter $t=4\pi$ in $\vec{r}_3(t)$

notice the difference here again!

In these cases, we see

$$C_1 = \{ \vec{r}_1(t) \; : \; 0 \le t \le 2\pi \}$$

$$= \{ \vec{r}_2(t) \; : \; 0 \le t \le \pi \}$$

$$= \{ \vec{r}_3(t) \; : \; 0 \le t \le 4\pi \}$$

All of these parameterizations represent the same curve C_1 with has:

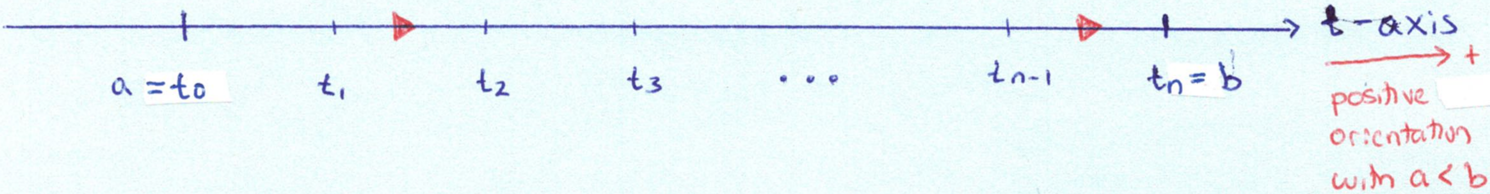
- ☐ starting point : $\langle 1, 0 \rangle$
- ☐ positive orientation : in counter clockwise "direction"
- ☐ end point : $\langle 1, 0 \rangle$
- ☐ Travels on the exact same path!

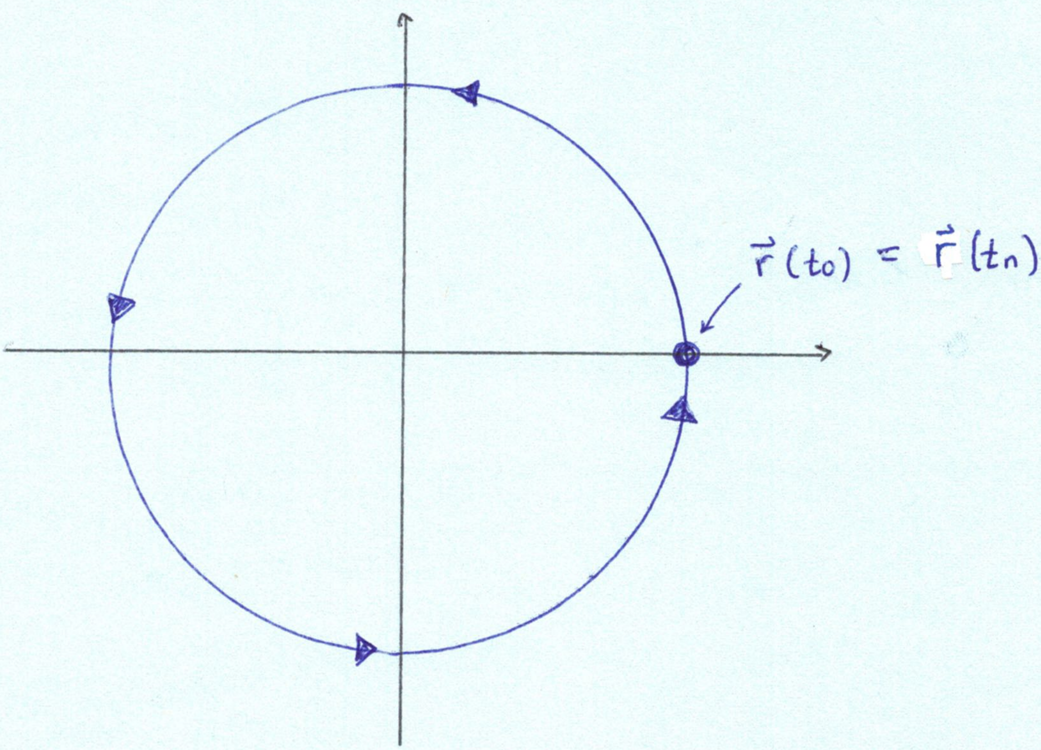
The major difference between each function is in how the parameter t relates to "distance" traveled along C_1.

Let's visualize the dynamics of encoding curve C

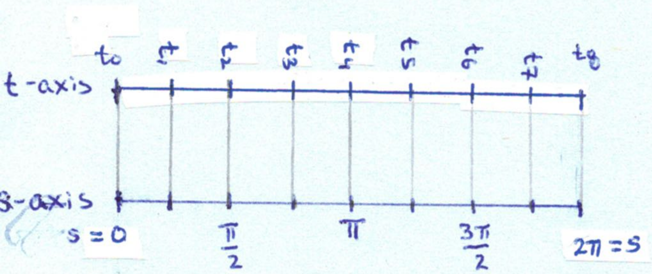
using a parameterized function $\vec{r} : D \subseteq \mathbb{R} \to \mathbb{R}^2$ with

$\vec{r}(t) = \langle x(t), y(t) \rangle$



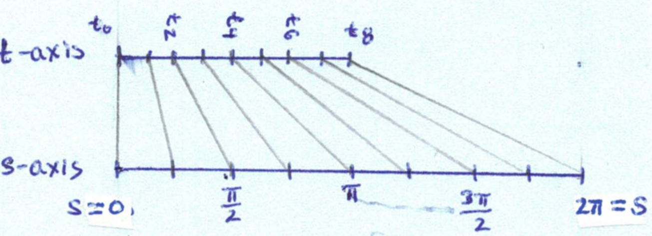


For the purposes of this thought experiment, let's set $n = 8$ and take a compare the parameter t to arc length s under each $\vec{r}_1(t)$, $\vec{r}_2(t)$, and $\vec{r}_3(t)$:



$D_1 = [0, 2\pi]$ $\quad \vec{r}_1 : D_1 \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$

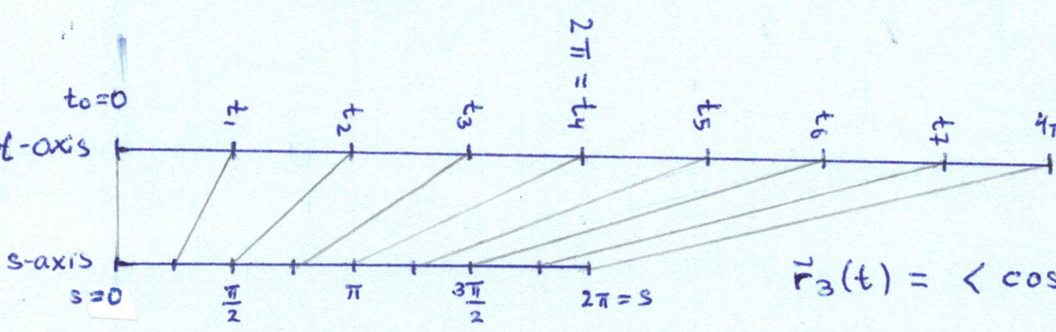
$\vec{r}_1(t) = \langle \cos(t), \sin(t) \rangle$



$D_2 = [0, \pi]$ $\quad \vec{r}_2 : D_2 \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$

$\vec{r}_2(t) = \langle \cos(2t), \sin(2t) \rangle$

$D_3 = [0, 4\pi]$, $\vec{r}_3 : D_3 \subseteq \mathbb{R} \rightarrow \mathbb{R}^2$



$\vec{r}_3(t) = \langle \cos(\frac{t}{2}), \sin(\frac{t}{2}) \rangle$

Notice, there are infinitely many ways to parameterize the curve C_1.

For a specific initial point and orientation, the arc length parameter is a "unique" parameter.

In other words, for all possible parametrization of C_1, there is only one for which

$$C_1 = \{ \vec{r}(s) : 0 \leq s \leq b \}$$

where s is the arclength

For our last example, this corresponds to the function $\vec{r}_1 : D_1 \subseteq \mathbb{R} \to \mathbb{R}^2$. The parameter t could have equivalently been called s for arc length.

The Arc Length Function

Let $C \subseteq \mathbb{R}^2$ be a smooth curve represented by the vector-valued function $\vec{r} : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ with

$$C = \{ \vec{r}(t) : t \in D \subseteq \mathbb{R} \}$$

and $\vec{r}(t) = \langle x(t), y(t) \rangle$. Of course, as t increases, the length we travel along C will also increase.

Using the arclength formula, the length of the portion of the curve C when traveling from point $\vec{r}(a)$ to point $\vec{r}(t)$ is given by

$$s(t) = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2}\, du$$

$$= \int_a^t \|\vec{r}'(u)\|_2 \, du$$

$$= \int_a^t \|\vec{v}(u)\|_2 \, du \qquad \text{where } \vec{v}(u) = \frac{d}{dt}\left[\vec{r}(u)\right]$$

This equation

$$s(t) = \int_a^t \| \vec{r}\,'(u) \|_2 \; du$$

gives an explicit relationship between the arc length $s(t)$ of the portion of curve C_1 connecting points $\vec{r}(a)$ and $\vec{r}(t)$ AND the parameter t used to describe the function $\vec{r}(t)$ that encodes C_1.

Let's use the fundamental theorem of calculus to consider

$$\begin{aligned} s'(t) &= \frac{d}{dt}\left[s(t) \right] \\ &= \frac{d}{dt}\left[\int_a^t \| \vec{r}\,'(u) \|_2 \; du \right] \\ &= \| \vec{r}\,'(t) \|_2 \qquad \text{by the fundamental thm of calculus} \\ &= \| \vec{v}(t) \|_2 \end{aligned}$$

Now, let's interpret this result. If parameter t represents time and $\vec{r}(t)$ is a function $\vec{r} : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ that gives the position of an object along a curve at time t, then the rate of change of the arc length w/r to time

$$\frac{d}{dt}\left[s(t)\right] = \frac{d}{dt}\left[\int_a^t \|\vec{r}\,'(u)\|_2 \, du\right] = \|\vec{r}\,'(t)\|_2$$

is the speed of the object at time t. We should notice

□ if $\vec{r}(t)$ is a smooth curve, then $\|\vec{r}\,'(t)\|_2 \neq 0$

$\Rightarrow \quad \|\vec{r}\,'(t)\|_2 > 0$

$\Rightarrow \quad \dfrac{ds}{dt} > 0$

$\Rightarrow \quad s(t) = \displaystyle\int_a^t \|\vec{r}\,'(u)\|_2 \, du$ is an increasing function

$\Rightarrow$ as t increases, then $s(t)$ increases

If $\vec{r}(t)$ is parameterized such that $\|\vec{r}\,'(t)\|_2 = 1$,

then

$$s(t) = \int_a^t \|\vec{r}\,'(u)\|_2 \, du$$

$$= \int_a^t 1 \, du$$

$$= t - a$$

which means the parameter t corresponds to arc length

$$\Rightarrow \vec{r}(t) = \vec{r}(s)$$

where we think of s as the "distance" traveled along curve C.

Example 13.8.6. p. 838 Arc length parametrization

Consider the helix $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 4t \rangle$

where $t \geq 0$.

A. Find the arc length function $s(t)$

B. Reparametrize the helix using arclength s as paramet

Solution: Recall that the arc length function is given by

$$s(t) = \int_0^t \|\vec{r}\,'(u)\|_2 \, du$$

We see, in this case, that

$$\vec{r}\,'(t) = \frac{d}{dt}\left[\langle 2 \cdot \cos(t), 2 \cdot \sin(t), 4t \rangle\right]$$

$$= \langle -2 \cdot \sin(t), 2\cos(t), 4 \rangle$$

Then, we see

$$\|\vec{r}\,'(t)\|_2^2 = (-2 \cdot \sin(t))^2 + (2 \cdot \cos(t))^2 + (4)^2$$

$$= 4 \sin^2(t) + 4 \cos^2(t) + 16$$

$$= 4 (\sin^2(t) + \cos^2(t)) + 16$$

$$= 4 + 16$$

$$= 20$$

$$\Rightarrow \quad s(t) = \int_0^t \sqrt{20}\, du = \int_0^t 2\sqrt{5}\, du$$

$\Rightarrow$ if we set $t = \dfrac{s}{2\sqrt{5}}$ into $\vec{r} : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$,

we get $\vec{r}_1(s) = \left\langle 2 \cdot \cos\left(\frac{s}{2\sqrt{5}}\right), 2 \sin\left(\frac{s}{2\sqrt{5}}\right), \frac{2s}{\sqrt{5}} \right\rangle$

which is our desired parameterization in terms of arc length.