Lesson 0: Introduction to Integral Theorems of Vector Analysis

Math 1AB Story line

Single-variable calculus focuses on the study of the ordinary derivative operator on single-variable functions

$$\underset{\text{Domain}}{F : D \subseteq \mathbb{R}} \longrightarrow \underset{\text{Codomain}}{\mathbb{R}}$$

Math 1A : Forward Problem of Ordinary Differentiation

$$\frac{d}{dx}\left[F(x) \right] = f(x) = F'(x)$$

ordinary derivative operator

Known single-variable function

unknown and desired derivative function

Math 1B: Backward Problem of Ordinary Anti-differentiation

$$\frac{d}{dx}\left[F(x) \right] = f(x)$$

ordinary derivative operator

unknown and desired "anti"-derivative function

Known derivative function

Example 1:

Find the following antiderivative

$$\int \cos(x)\,dx$$

differential form

integral sign

integrand

Note: Since there are no limits of integration, we classify this as indefinite integra

Solution:

Let $f(x) = \cos(x)$.

$$F(x) = \int \cos(x) \quad dx$$

$$\Rightarrow \quad \frac{d}{dx}\left[F(x)\right] = f(x) = \cos(x)$$

$$\Rightarrow \quad F(x) = \sin(x) + C$$

Note that when "solving" an indefinite integral, our final "answer" is an entire class of functions.

$$\int \cos(x)\, dx = \sin(x) + c$$

$$= \{ \sin(x) + c \; : \; c \in \mathbb{R} \}$$

this is a set of functions

In other words, for indefinite integrals the integral acts as an operator

$$\int [f(x)]\, dx = F(x) = A(x) + C$$

input a function

output a class of functions

LO,P

Math 1B: Backward Problem of

Ordinary Antidifferentiation

The fundamental theorem of Calculus

Let $f: D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be

a continuous function on $D = [a,b]$.

Then the area function

upper bound is variable

$$\underbrace{A(x)}_{\text{this is a function of variable } x} = \int_a^{x} f(t)\, dt$$

is continuous on $[a,b]$ and differentiable on (a,b). Moreover

$A'(x) = \frac{d}{dx}\left[A(x)\right]$ has very special properties.

LO, P

$$A'(x) = \frac{d}{dx}\left[A(x)\right]$$

$$= \frac{d}{dx}\left[\int_t^x f(t)\,dt\right]$$

$$= f(x)$$

$\Rightarrow$ $A(x)$ is the antiderivative of $f(x)$ on $[a, b]$

$$\Rightarrow \quad \frac{d}{dx}\left[A(x)\right] = f(x)$$

"indefinite integral produces this Antiderivative function"

Thus, the "solution" to an antiderivative problem is a class of functions

$$\underbrace{F(x)}_{\text{general solution}} = \underbrace{A(x)}_{\text{area-under the curve function}} + c$$

any constant since $\frac{d}{dx}[c] = 0$

general solution called an antiderivative

area-under the curve function

Example 2: Find the area under the curve

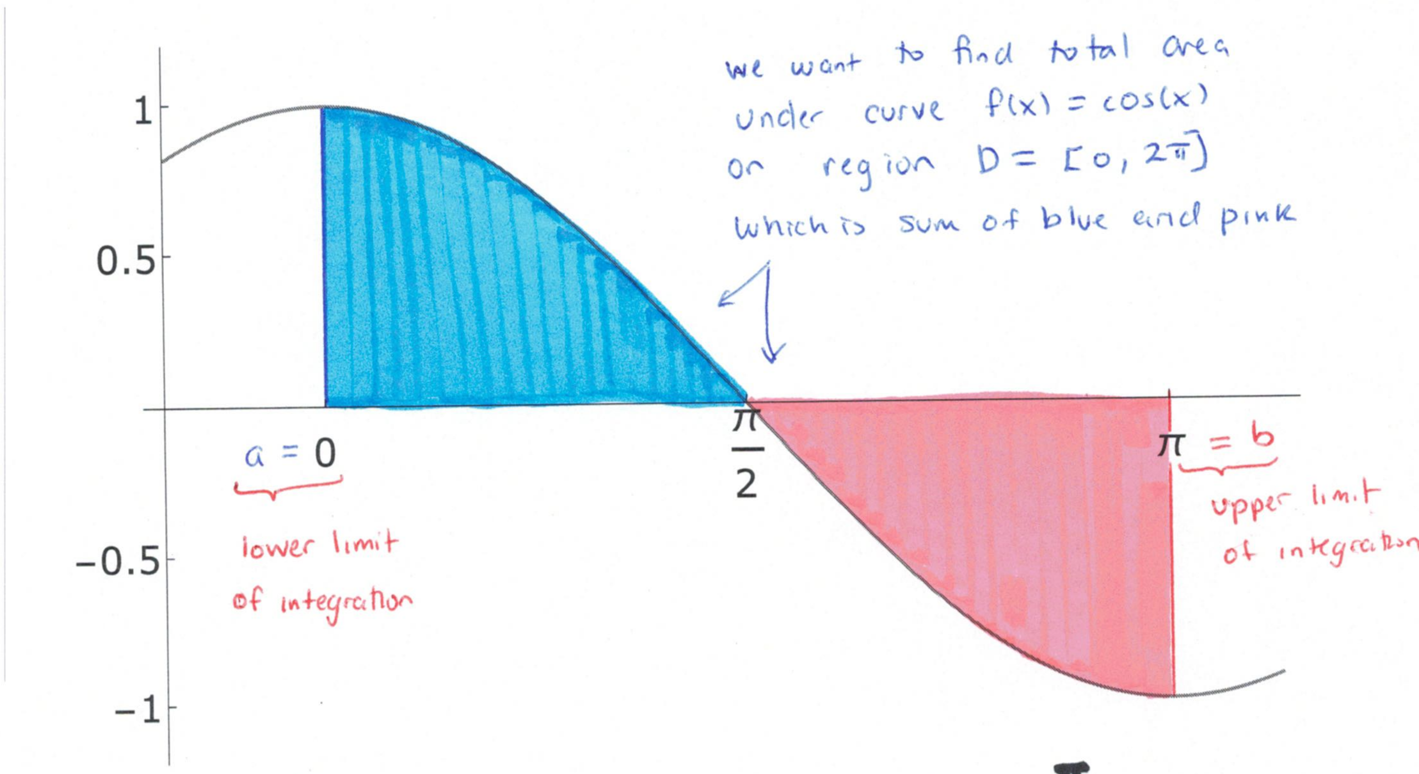
upper limit of integration

$$\int_0^{\pi} \underbrace{\cos(x)}_{\text{integrand}} \, dx \longleftarrow \text{differential form}$$

lower limit of integration

This is a <u>definite</u> integral! The "answer" is a scalar representing area under a curve.

Solution: Let's begin by graphing the integrand on the region $D = [0, \pi]$ as seen on the following page.



Using this graph we guess that $\int_0^{\pi} \cos(x)\,dx = 0$.

We confirm this via the Fundamental thm of calculus:

$$\int_0^{\pi} \cos(x)\,dx = \sin(x) \Big|_0^{\pi}$$

$$= \sin(\pi) - \sin(0)$$

$$= 0 - 0 = \boxed{0.}$$ ✓

Example 3: Find the area under the curve

upper limit of integration

integral sign

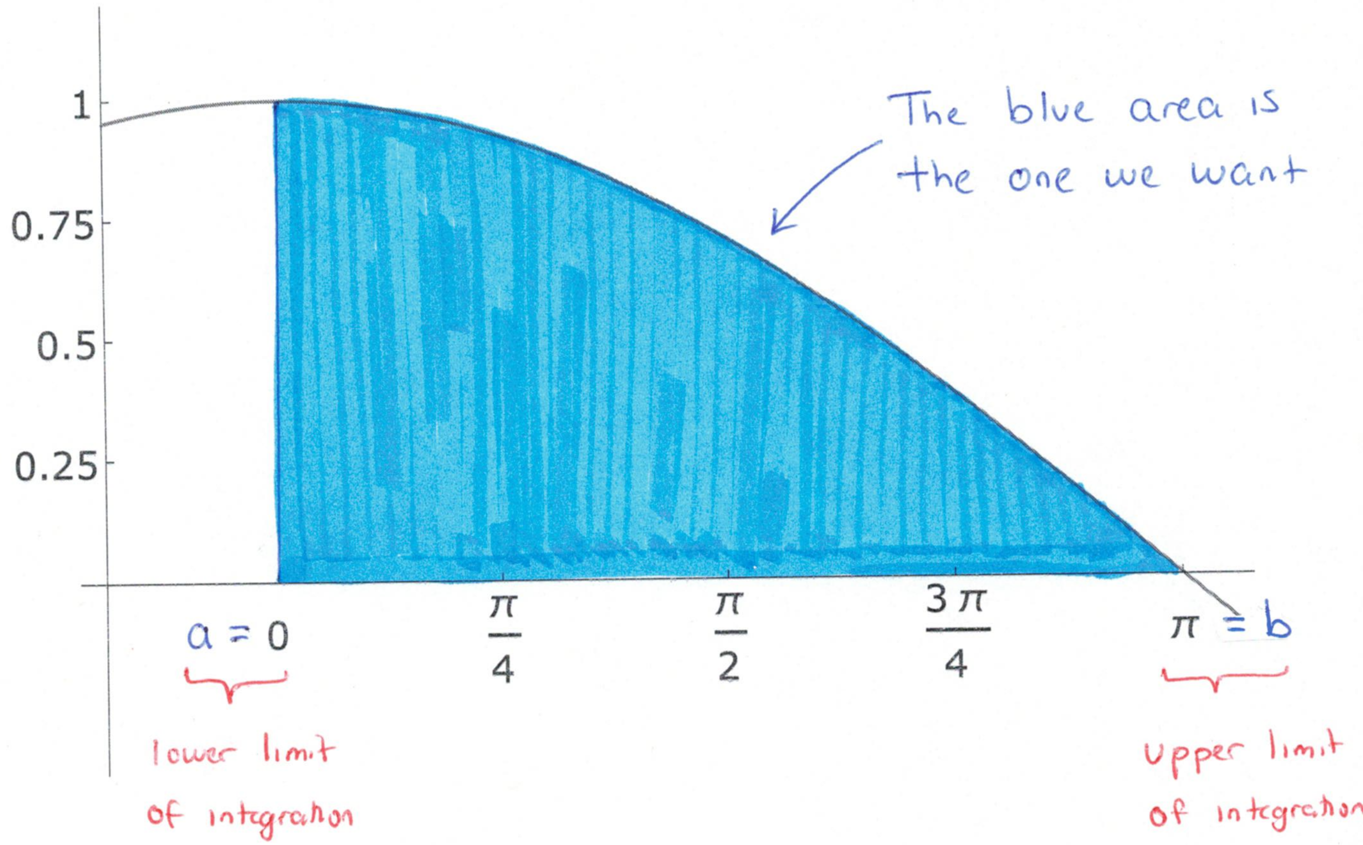
$$\int_0^{\pi} \underbrace{\cos\left(\frac{t}{2}\right)}_{\text{integrand}} \, dt$$

differential form

lower limit of integration

Note: the moment we see the limits of integration, we classify this as a definite integral... Moreover, we make some subtle assumptions to empower analysis.

Solution: Let's begin with a graph:



Using the fundamental theorem of calculus, we know

$$\int_0^{\pi} \cos\left(\frac{t}{2}\right) dt = 2 \sin\left(\frac{t}{2}\right)\Bigg|_0^{\pi}$$

$$= 2 \cdot \sin\left(\pi/2\right) - 2 \sin(0) = \boxed{2}$$

$\boxed{\text{LO, p12}}$

You might have noticed a problem with this definite integral that requires us to think slowly:

upper limit of integration

(parameterized) input to integrand

$$\int_0^{\pi} \underbrace{\cos\left(\frac{t}{2}\right)}_{\text{integrand}} \, dt$$

(nonparameterized) differential form

lower limit of integration

(nonparameterized) pure input to cosine that is different than differential form

If we set $f(x) = \cos(x)$, then we see the cosine function is the integrand.

In the easy case $\int \cos(x)\, dx = \sin(x)$

However, our life is not so easy.

Rewriting this integral, we have

the input to cosine is not a "pure" t variable

$$\int_{\text{"}t=0\text{"}}^{\text{"}t=\pi\text{"}} \cos\left(\frac{t}{2}\right)\, dt$$

the differential form dt suggests we want to work with "pure t" variables ...

the limits of integration are assumed to be "purely" in t (nonparameterized)

As we will see, I will beg you to think about the differential form as suggesting a mechanism to measure "sizes" (or weights) of intervals in the input domain space of integral. Here the problem is that the suggested size recommendation is not aligned exactly with input to Inte

How do we deal with this?

Consider the following definite integral

upper limit of integration

$$\int_{0}^{\pi} \cos\left(\frac{t}{2}\right) dt = \int_{\text{"}t=0\text{"}}^{\text{"}t=\pi\text{"}} \cos\left(\frac{1}{2}\cdot t\right) dt$$

lower limit of integration (assumed to be with respect to t)

"U"-substitution

Let $x(t) = \frac{t}{2}$

$\Rightarrow dx = \frac{1}{2} dt$

$\Rightarrow dt = 2\,dx$

and we know:

$t = \pi \Rightarrow x(t) = x(\pi) = \pi/2$

$t = 0 \Rightarrow x(t) = x(0) = 0$

$$= \int_{\text{"}x=0\text{"}}^{\text{"}x=\pi/2\text{"}} \cos(x) \cdot 2\,dx$$

$$= \int_{0}^{\pi/2} 2 \cdot \cos(x)\, dx$$

$$= 2 \sin(x) \Big|_{0}^{\pi/2} = 2\left(\sin(\pi/2) - \sin(0)\right) = \boxed{2}$$

LO, P15

WARNING: When I was in "Math 1B" at UCSB, I remember my teacher told me: "when doing u-substitut[ion]

you can use the following steps

$$\int_0^{\pi} \cos(t/2)\, dt$$

Let $x(t) = \frac{t}{2}$

$\Rightarrow \frac{dx}{dt} = \frac{1}{2}$

$\Rightarrow dx = \frac{1}{2}\, dt$

$$\frac{dx}{dt} \cdot dt = \frac{1}{2} \cdot dt$$

as if I can "multiply" both sides by dt.

NO

this is absurd! dt is not a real number and thus the intuition in $\mathbb{R}$ does not apply rigorously here!!

If we do this, make sure to track and appropriately account for values of integral limits.

That advice was complete and utter nonsense designed to get me through Math 1B with as little friction as possible (even at the expense of deep understanding).

Indeed: this is bull shit math

NO! Illegal (calbiet intuitive)

$$\frac{dx}{dt} = \frac{1}{2} \Rightarrow \frac{dx}{dt} \cdot dt = \frac{1}{2} \cdot dt$$

$$\Rightarrow dx = \frac{1}{2} dt$$

NO! Not like this

(this last line is what we want... the steps together are not valid)

Instead, as we work together, I will be encouraging you to develop intuition about the differential forms we use in all integrals.

For now, I will encourage you to analyze the notation more carefully.

Anatomy of an ordinary integral

upper limit of integration

differential form

$$\int_a^b f(x)\,dx$$

integrand

lower limit of integration

Note on the "dx" notation:

There are a few useful ways to think about this notation:

- referred to as a <u>differential form</u>
- dx indicates we are integrating with respect to x: when we perform our limiting process

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$

we measure the size of intervals Δx_k along a path on x-axis.

- delimits the integrand (indicates where the integrand ends)

- I want you to think about this as a symbol to represent "sizes" in domain

In other words, when we see

integrand

integral symbol

$$\int_D f \, d\boxed{}$$

differential form

region in domain over which we will integrate

where $\displaystyle f: \underbrace{D \subseteq \mathbb{R}^n}_{\substack{\text{multiple} \\ \text{variable} \\ \text{(input)}}} \longrightarrow \underbrace{\mathbb{R}^m}$

if $m = 1$: real-valued output

if $m > 1$: vector-valued output

$\underbrace{\text{the symbol immediately after the "d"}}$

$$d\boxed{}$$

will encode lots of juicy info. A major part of your job in this class is to interpret this symbol accurately

LO,P2

In our simple case, think of dt in

$$\int_0^{\pi} \cos\left(\frac{t}{2}\right) dt = \int_D f \, dt$$

as a function $t = t($input to integral$)$

defined only on the geometric object

that creates the domain of integrand

here is function t

$$\cos(x) = \cos\left(\frac{t}{2}\right)$$

here is input to integral

We want these to "align": $x = \frac{t}{2}$

$$\Rightarrow t(x) = 2 \cdot x$$

This function $t: D \longrightarrow \mathbb{R}$

and the corresponding differential form

notation

$$d\boxed{t} = 2\,dx$$

measuring domain region in t suggests a region twice as large as measuring in x

tell us how we should measure the "size" (or weight) of objects in the domain region D.

Habit: Spend at least 60 seconds every time you see the "d"

$$d\boxed{}$$

asking yourself: what does this mean about "size" measure

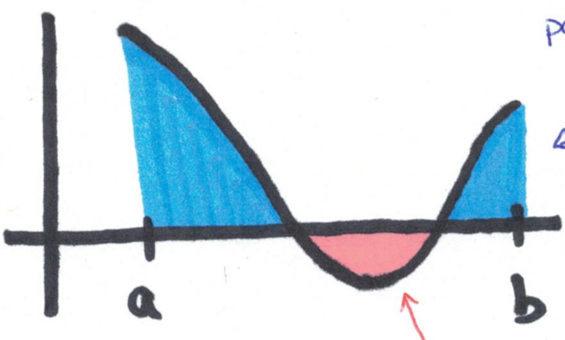
Math 1B: Area under the Curve

Definite Integrals

In Math 1B we studied a related problem known as definite integration to find "area under a curve" in the form

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Net area "under" curve}} \in \mathbb{R}$$

← the "solution" to such a problem is a signed scalar representing an area



Blue represents positive (+) area

Red represents negative area

In order to construct the single-variable definite integral, we start by studying the problem of computing area "under" a curve. To do so, we let $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function on the interval

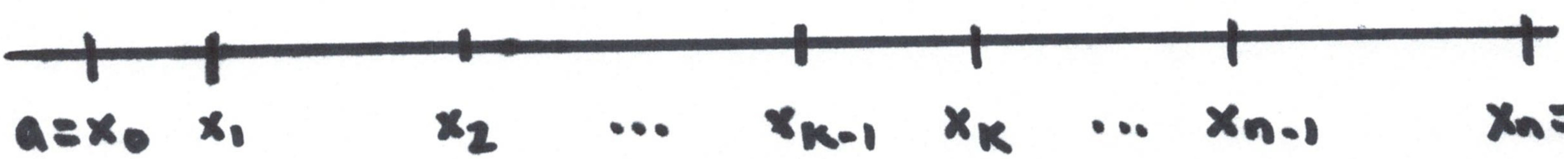
$$D = [a, b] \subseteq \mathbb{R}$$

where $a, b \in \mathbb{R}$ with $a < b$.

In order to create a finite approximation for our area, we begin by constructing a **general** finite partition of the domain

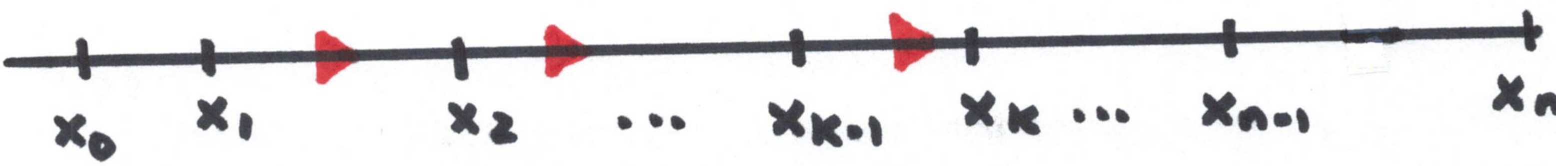
$$D = [a, b]$$

into n subintervals that are not necessarily equal in "length":



Since we read left to right, we suppose our orientation is such that

$$\underbrace{a = x_0}_{\text{left endpoint of } D} < x_1 < x_2 \cdots < x_{n-1} < \underbrace{x_n}_{\text{right endpoint of } D} = b$$



orientation : $- \longrightarrow +$ positive toward the "right"

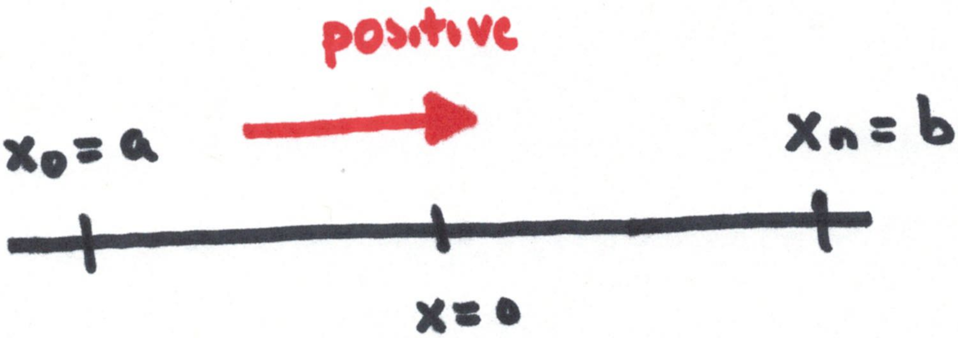
In other words, to take a definite integral, the domain space $D = [a,b]$ is assumed to be <u>oriented</u> (we will return to this idea throught the course).

In the simplist case of a domain

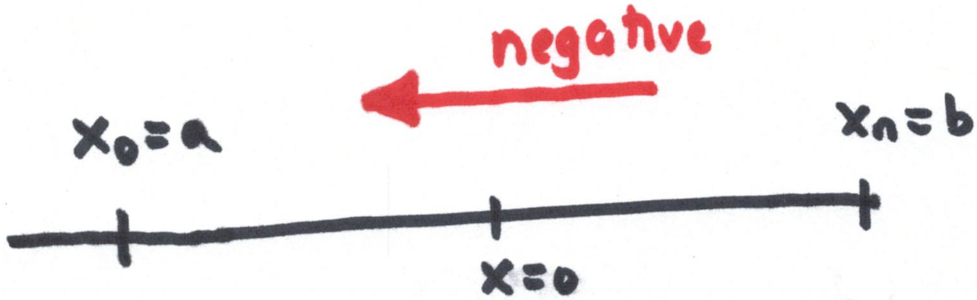
$$D = [a,b] \subseteq \mathbb{R}$$

the orientation represents a mechanism to assign positive signs or negative signs with reference to a path traveled

positive orientation



negative orientation

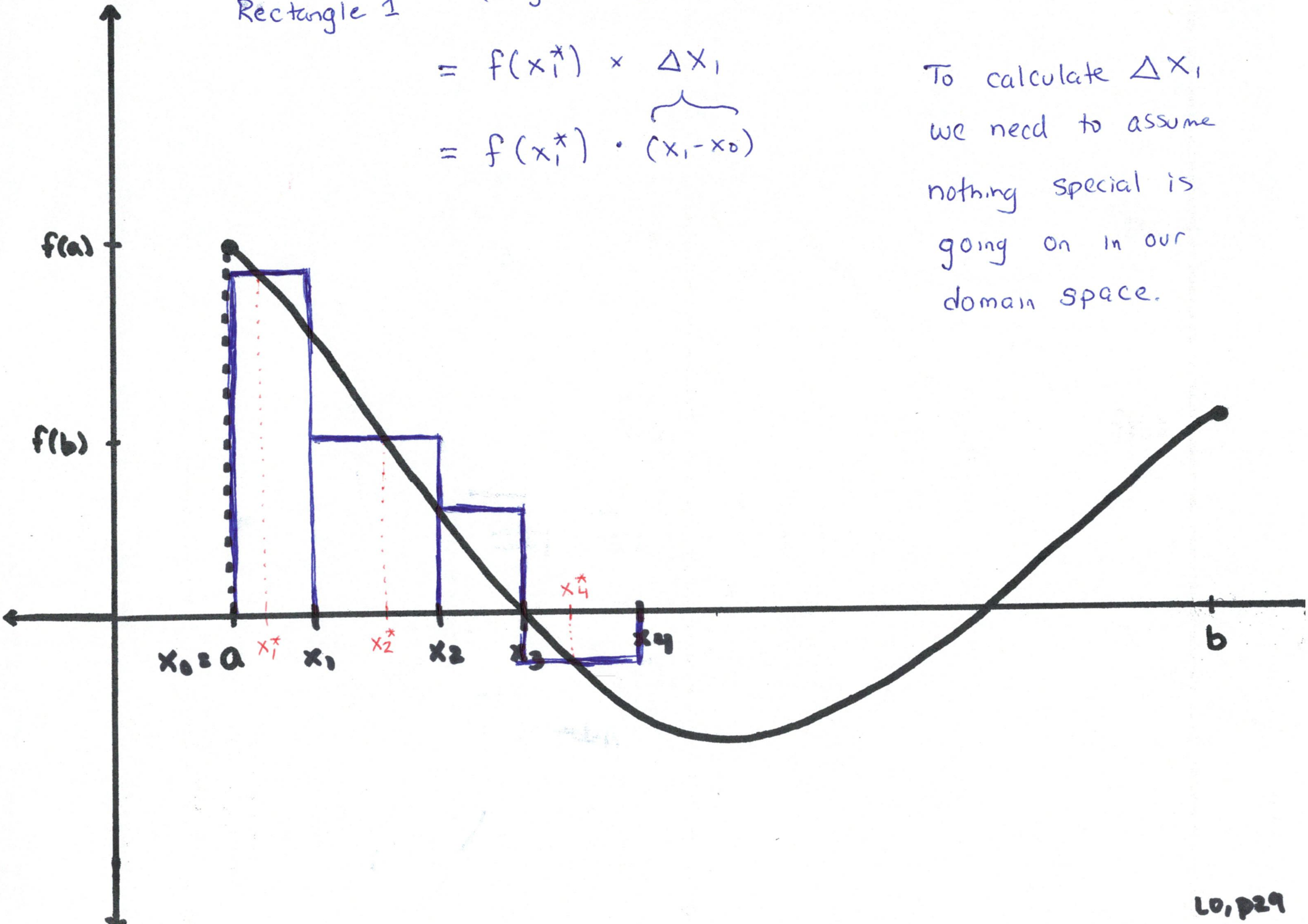


Area of Rectangle 1 $= (\text{height}) \times (\text{interval size})$

$= f(x_1^*) \times \Delta x_1$

$= f(x_1^*) \cdot \overbrace{(x_1 - x_0)}$

To calculate Δx_1 we need to assume nothing special is going on in our domain space.



In this case, the subintervals are given by

1st subinterval: $[x_0, x_1]$

2nd subinterval: $[x_1, x_2]$

$\vdots$

Kth subinterval: $[x_{K-1}, x_K]$

$\vdots$

n th subinterval: $[x_{n-1}, x_n]$

Notice, we make the assumption that the domain set is oriented (ordered) in a very special way

To create a finite approximation for the area under the curve, we choose a representative point x_k^* in the kth subinterval

$$x_k^* \in [x_{k-1}, x_k]$$

Then, we sample the "height" of the curve at x_k^* and multiply by the "oriented weight (size) of the subinterval

Then, the definite integral is a limit calculated as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*) \cdot \Delta x_k$$

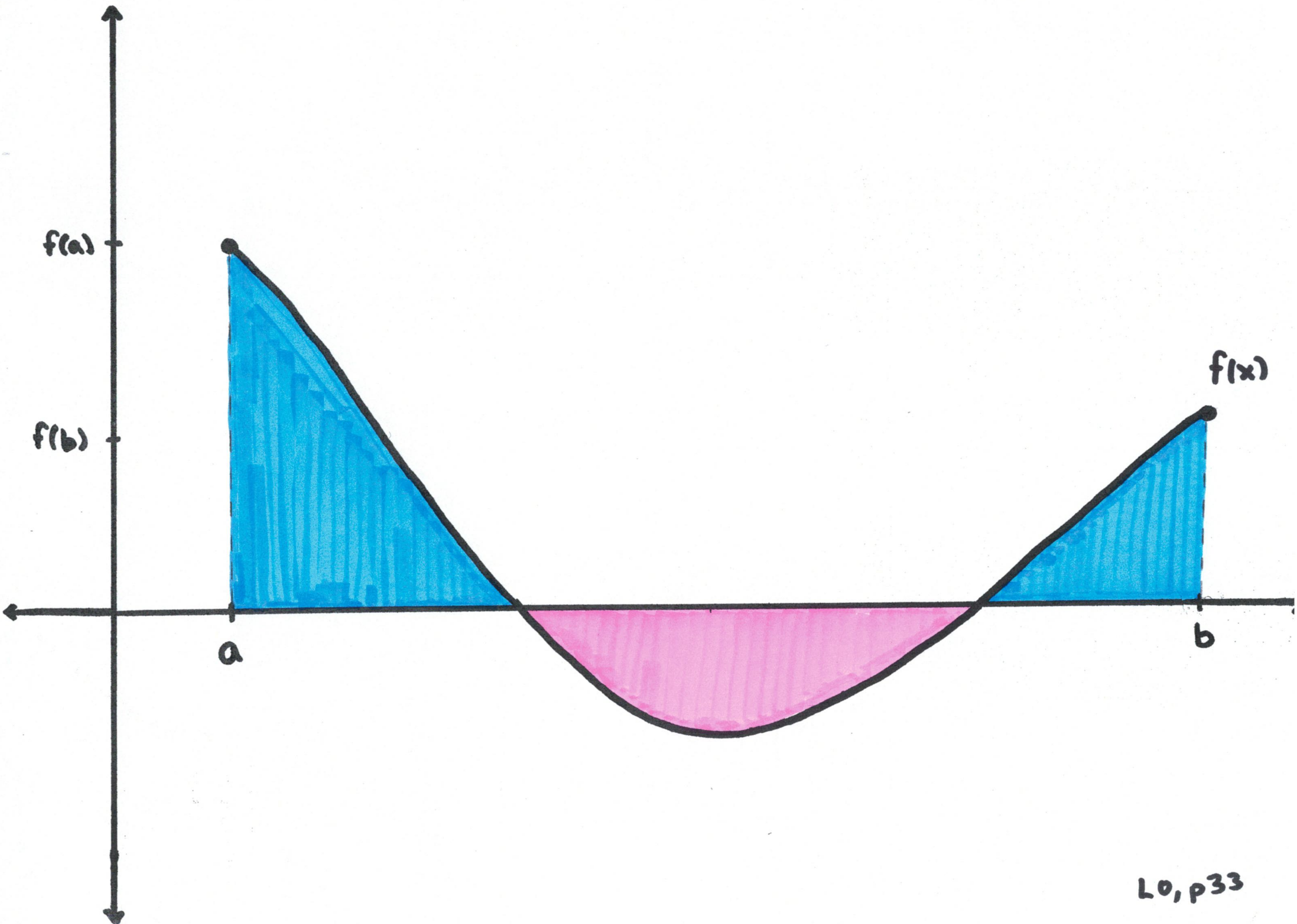
One underlying assumption is that endpoints stay fixed with

$x_0 = a$

$x_n = b$

$$= \lim_{\Delta\to 0} \sum_{k=1}^{n} f(x_k^*) \cdot \Delta_k$$

where $\Delta = \max_{k\in\mathbb{N}} \Delta x_k$



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