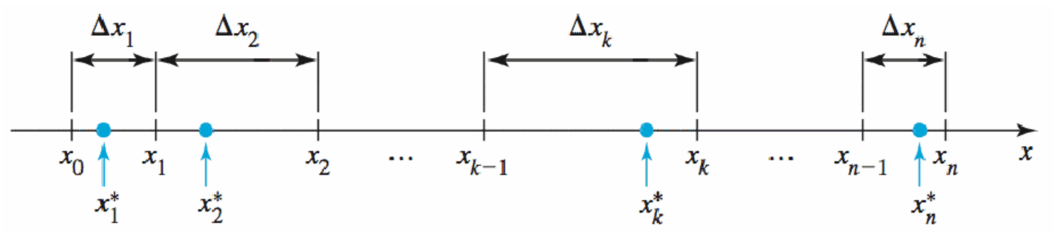
Lesson 0: Handout
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Chapter 5: Integration, p. 333 -397

Definition. *p. 351* ***General Riemann Sum on*** $D = [a, b] \subseteq \mathbb{R}$

Suppose that $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{n-1}, x_n]$ are subintervals of $[a, b]$ with

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Let Δ_k be the "length" of the subinterval $[x_{k-1}, x_k]$. For this definition, assume that $\Delta_k = x_k - x_{k-1}$. Let $x_k^* \in [x_{k-1}, x_k]$ be any point in the kth subinterval, for $k = 1, 2, ..., n$. We can visualize this set up on the $x-$axis as follows:



If $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is a single-variable, real-valued function defined on $D = [a, b]$, then the finite sum given by

$$\sum_{k=1}^{n} f\left(x_k^*\right) \Delta_k = f\left(x_1^*\right) \Delta_1 + f\left(x_2^*\right) \Delta_2 + \cdots + f\left(x_n^*\right) \Delta_n$$

is called the **general Riemann sum for** f **on** $D = [a, b]$

In the definition to follow, we introduce some subtle notation in the limit. In particular, we set

$$\Delta = \max\{\Delta_1, \Delta_2, ..., \Delta_n\}$$

If we take the limit $\Delta \to 0$, then we conclude that $\Delta_k \to 0$. With this in mind, we can define integrability.

Definition. *p. 351* ***Definite integral over*** $D = [a, b] \subseteq \mathbb{R}$

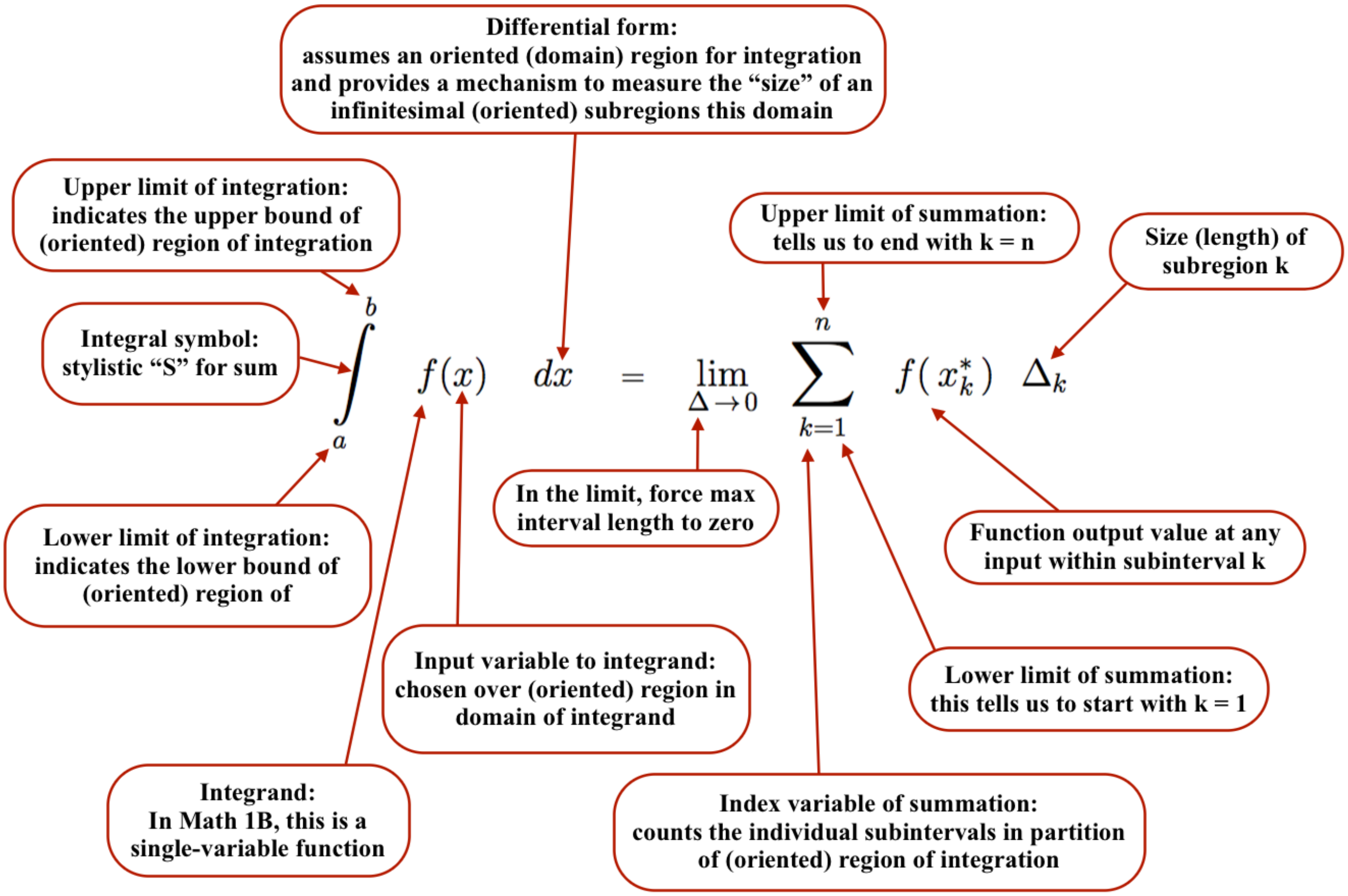
Assume we have the same set up from our definition of the general Riemann sum for f on $[a, b]$. If the following limit

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta_k$$

exists and is unique for all partitions of $[a, b]$ and all choices of $x_k^* \in [x_{k-1}, x_k]$ on a partition, then we say that the function f is **integrable** on $[a, b]$. The limit is the **definite integral of** f **on** $[a, b]$, which we write

$$\int_a^b f(x) \quad dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta_k$$

Let's take a look at the notation we use to represent a definite integral. In particular, we identify each of the individual components of this notation and delineate the underlying assumption behind these components. As you might notice, the definite integral notation compresses many important assumptions and hides subtle meaning within notation. This analysis will lead to some interesting observations later in Math 1D about the use of integral notation to represent the ideas behind the multivariable integration theorems we study in this class.



In the case of general Riemann sums, we take our region of integration $D = [a, b]$ and establish a general partition of this region into n subintervals

$$[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$$

with $x_0 = a$, $x_n = b$ and

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

In the general case, we denote the "size" of the kth subinterval $[x_{k-1}, x_k]$ by Δ_k. In the (simplest) case of Riemann integration, we measure the "size" of each subinterval as the "length" of that subinterval with

$$\Delta_k = x_k - x_{k-1}$$

for $k = 1, 2, ..., n$. Because a general partition does not require a uniform discretization of the region $[a, b]$, each of the k subintervals can have a different length. Moreover, when we sample the function, we let x_k^* be any point on the subinterval $[x_{k-1}, x_k]$ without requiring a specific pattern. We can contrast the general case with the more specific regular partitions we use when taking Riemann integrals using the left-hand rule, the right hand rule, the midpoint rule, the trapezoid rule, or Simpson's rule.

Definition. *p. 336* ***Regular partition (uniform discritization)***

Suppose $[a, b]$ is a closed interval in $\mathbb{R}$ containing n subintervals

$$[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n]$$

of equal "size" given by

$$\Delta x = \frac{b-a}{n}$$

with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, x_2, ..., x_{n_1}, x_n$ of the subintervals are called uniform **grid points** and they create a **regular partition** (uniform discritization) of the interval $[a, b]$. In general under these assumptions, the kth grid point is

$$x_k = a + k\,\Delta x$$

for $k = 0, 1, 2, ..., n$.

Definition. *p. 337* ***Special Riemann Sums***

Suppose $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ is a single-variable, real-valued function. Suppose the domain $[a, b]$ is divided into n subintervals of equal length Δx. If x_k^* is any point in the kth subinterval $[x_{k-1}, x_k]$, for $k = 1, 2, ..., n$, then

$$f\left(x_1^*\right)\Delta x + f\left(x_2^*\right)\Delta x + \cdots + f\left(x_n^*\right)\Delta x$$

is called a **Riemann sum** for f on $[a, b]$. This sum is called a Riemann sum computed via:

i. the **left-hand rule** if $x_k^* = x_{k-1}$ is the left endpoint of $[x_{k-1}, x_k]$.

ii. the **right-hand rule** if $x_k^* = x_k$ is the right endpoint of $[x_{k-1}, x_k]$

iii. the **midpoint rule** if $x_k^* = \dfrac{x_k - x_{k-1}}{2}$ is the midpoint of $[x_{k-1}, x_k]$