---

1. Consider the single-variable, vector-valued function:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle$$

Suppose that the functions $x(t)$ and $y(t)$ have continuous derivatives for all $t \in [a, b]$. Let's construct the limit definition of the derivative $\mathbf{r}'(t)$.

A. (6 points) Find two points on the curve C, call them P_0 and P. Construct the vector-valued equation for the secant line through these two points.

Solution: Let $P_0(x_0, y_0) = \mathbf{r}(t_0)$ and $P(x, y) = \mathbf{r}(t)$ be two points on the curve C where $t_0, t \in [a, b]$. Then, we can construct the vector $\mathbf{v} \in \mathbb{R}^2$ with a tail at point P_0 and head at point P as

$$\mathbf{v} = \mathbf{r}(t) - \mathbf{r}(t_0) = \langle x - x_0, y - y_0 \rangle$$

The secant line through these two points is given by

$$\ell(\alpha) = \mathbf{p}_0 + \alpha \cdot \mathbf{v}$$

where $\alpha \in \mathbb{R}$ is a parameter.

B. (6 points) Using the proper limit, construct the derivative vector $\mathbf{r}'(t)$ as the vector that defines the direction of the tangent line. Also, explain how we know that

$$\mathbf{r}'(t) = \langle\ x'(t)\ ,\ y'(t)\ \rangle$$

Solution: To define the "direction" of the tangent vector to the curve C at the point $\mathbf{r}(t_0)$, we will take two steps. First, we recognize that the tangent vector to the curve C will be parallel to the vector $\mathbf{v} = \mathbf{r}(t) - \mathbf{r}(t_0)$ when we take the limits as $t \longrightarrow t_0$. However, the problem is that when we attempt to find the limit

$$\lim_{t \to t_0} \mathbf{r}(t) - \mathbf{r}(t_0) = 0$$

we see that the limit approaches the zero vector, since $\mathbf{r}(t)$ converges to the point $\mathbf{r}(t_0)$ in the limit. However, we can define the rescaled tangent vector as

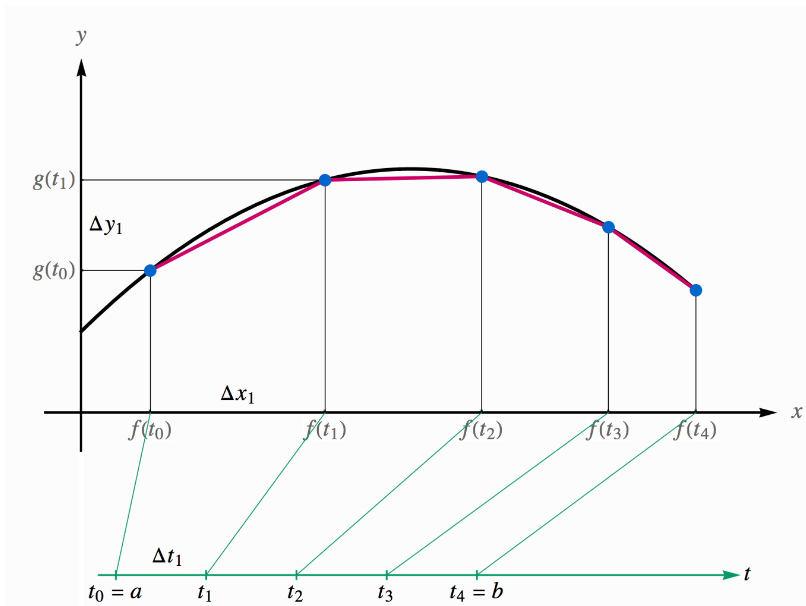
$$\begin{aligned} \mathbf{r}'(t_0) &= \lim_{t \to t_0} \frac{1}{t - t_0} \cdot (\mathbf{r}(t) - \mathbf{r}(t_0)) \\ &= \lim_{t \to t_0} \left\langle \frac{x - x_0}{t - t_0}, \frac{y - y_0}{t - t_0} \right\rangle \\ &= \left\langle \lim_{t \to t_0} \frac{x - x_0}{t - t_0}, \lim_{t \to t_0} \frac{y - y_0}{t - t_0} \right\rangle \\ &= \langle x'(t_0), y'(t_0) \rangle \end{aligned}$$

2. Suppose that we have a parameterized curve $C = \{\, \mathbf{r}(t) \,:\, a \le t \le b \,\}$ where $\mathbf{r}(t) = \langle x(t), y(t) \rangle$. Suppose that the functions $x(t)$ and $y(t)$ have continuous derivatives for all $t \in [a, b]$.

A. (6 points) Let L represent the total arc length of the curve C. Using the diagram below, explain why we can approximate L, the arc length of the curve C, as

$$L \approx \sum_{k=1}^{n} \sqrt[2]{\,(\Delta x_k)^2 + (\Delta y_k)^2\,}$$

Notice that in the diagram, we assume that $n = 4$.



Solution: Please see Lesson 10: Jeff's Handwritten Notes.

B. (6 points) Explain how to use the mean value theorem and a limit to derive the precise arc length formula

$$L = \int_a^b \|\mathbf{r}'(t)\|_2 \; dt$$

Make sure to explain how the limit affects each term in the approximation and results in the stated definition for calculating the arc length L of the curve C.

Solution: Please see Lesson 10: Jeff's Handwritten Notes.

© Jeffrey A. Anderson

3. (8 points) Evaluate the integral:

$$\iiint_D e^{\left(x^2+y^2+z^2\right)^{3/2}} \, dV$$

where D is the unit ball in $\mathbb{R}^3$ centered at the origin. (Hint: try a change of variables into spherical coordinates).

Solution: It is worth noting that if we try to integrate this function as written using iterated integrals, we very quickly fail. Based on our hint, we will try change of variables to spherical coordinates. We recall that when transforming from cartesian coordinates into spherical coordinates, we have

$$\rho^2 = x^2 + y^2 + z^2$$

Moreover, by the description of our region given in the problem, we know that

$$D = \{ (\rho, \phi, \theta) \, : \, 0 \le \rho \le 1, \, 0 \le \phi \le \pi, \, 0 \le \theta \le 2\pi \}$$

Finally, we remember from our derivation that the differential form translation from cartesian to spherical coordinates takes the form:

$$dV = dx\, dy\, dz = \rho^2 \, \sin(\phi) \, d\rho \, d\phi \, d\theta$$

Then, let's take our triple integral using iterated integration in spherical coordinates:

$$\begin{aligned}
\iiint_D e^{\left(x^2+y^2+z^2\right)^{3/2}} \, dV &= \int_0^1 \int_0^\pi \int_0^{2\pi} e^{\left(\rho^2\right)^{3/2}} \, \rho^2 \sin(\phi) \, d\theta \, d\phi \, d\rho \\
&= \int_0^1 \int_0^\pi \int_0^{2\pi} \rho^2 \cdot e^{\rho^3} \, \sin(\phi) \, d\theta \, d\phi \, d\rho \\
&= 2\pi \int_0^1 \int_0^\pi \rho^2 \cdot e^{\rho^3} \, \sin(\phi) \, d\phi \, d\rho \\
&= 2\pi \int_0^1 \rho^2 \cdot e^{\rho^3} \int_0^\pi \sin(\phi) \, d\phi \, d\rho \\
&= 2\pi \int_0^1 \rho^2 \cdot e^{\rho^3} \left(-\cos(\phi)\Big|_0^\pi \right) d\rho \\
&= 4\pi \int_0^1 \rho^2 \cdot e^{\rho^3} \, d\rho
\end{aligned}$$

© Jeffrey A. Anderson

Now we need to find this last integral. Notice that we can use a substitution technique to simplify our integral. In particular, we set

$$u = \rho^3 \qquad \Longrightarrow \qquad du = 3\,\rho^2.$$

We also see that $u(1) = 1$ and $\rho(0) = 0$. Now we do a change of variables.

$$\begin{aligned}\iiint\limits_D e^{\left(x^2+y^2+z^2\right)^{3/2}}\; dV &= \frac{4\pi}{3}\int\limits_0^1 e^{\rho^3}\; 3\,\rho^2\; d\rho \\ &= \frac{4\pi}{3}\int\limits_0^1 e^u\; du \\ &= \frac{4\pi}{3}\, e^u\Big|_0^1\end{aligned}$$

We conclude by evaluating this expression to find:

$$\iiint\limits_D e^{\left(x^2+y^2+z^2\right)^{3/2}}\; dV = \boxed{\frac{4\pi}{3}\,(e-1)}$$

4. (8 points) Compute the line integral:

$$\int_C f \ ds$$

where $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ is given by $f(x, y) = e^{x+y}$ and C is the line segment from the point $O(0, 0)$ to the point $P(2, 1)$

Solution: Let's begin by describing our curve via a parametric equation. Specifically, we have $C = \{\ell(t) \, : \, 0 \leq t \leq 1\}$ where

$$\ell(t) = \langle 0, 0\rangle + t \cdot \langle 2, 1\rangle = \langle \, 2\,t, \, t \, \rangle = \langle x(t), y(t)\rangle$$

We also remember from our work in class that since we define the arc length function as

$$s(t) = \int_a^t \| \, \ell\,' \, (u) \, \|_2 \ du$$

we know via the fundamental theorem of calculus that

$$s'(t) = \frac{d}{dt} \int_a^t \| \, \ell\,' \, (u) \, \|_2 \ du = \| \, \ell\,' \, (t) \, \|_2$$

By changing variables, we know $ds = \| \, \ell\,' \, (t) \, \|_2 \, dt$. With this in mind, we can calculate the line integral

$$\int_C f \ ds = \int_C f(x(t), y(t)) \ \| \, \ell\,' \, (t) \, \|_2 \, dt$$

$$= \int_0^1 e^{x(t)+y(t)} \ \sqrt{5} \, dt$$

$$= \sqrt{5} \int_0^1 e^{3t} \ dt$$

$$= \frac{\sqrt{5}}{3} \ e^{3t} \Big|_0^1$$

We see that our desired line integral is

$$\int_C f \ ds = \boxed{\frac{\sqrt{5}}{3} \ \left(e^3 - 1\right)}$$

5. (10 points) Compute the circulation of the vector field

$$F(x, y) = \langle 2y, -2x \rangle$$

along the unit circle C, oriented counterclockwise.

Solution: Let's begin by describing our curve via a parametric equation. Specifically, we have $C = \{\mathbf{r}(t) : 0 \le t \le 2\pi\}$ where

$$\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle = \langle x(t), y(t) \rangle$$

We recall that the circulation is given as the following line integral:

$$\int_C f \ ds = \oint_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} \, ds = \oint_C \vec{\mathbf{F}}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

Based on our problem statement, we can write

$$\vec{\mathbf{F}}(\mathbf{r}(t)) = \vec{\mathbf{F}}(x(t), y(t)) = \langle 2\,y(t) \, , \, -2\,x(t) \rangle = \langle 2 \sin(t) \, , \, -2 \cos(t) \rangle$$

We also have that

$$\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle = \langle -\sin(t), \cos(t) \rangle$$

We can use these two calculations to find our circulation on this curve:

$$\begin{aligned}
\int_C f \ ds &= \oint_C \vec{\mathbf{F}}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\
&= \oint_C \langle 2 \sin(t) \, , \, -2 \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \ dt \\
&= -\oint_0^{2\pi} 2\sin^2(t) + 2\cos^2(t) dt \\
&= -\oint_0^{2\pi} 2 dt \\
&= -4\pi
\end{aligned}$$

Optional Challenge Problem

6. (5 points) State and prove the fundamental theorem of calculus.

Solution: Below we state the Fundamental Theorem of Calculus:

Theorem 5.5. *p. 366* ***Fundamental Theorem of Calculus, Part 1***

Let $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined on the region $D = [a, b]$. Define the area function

$$A(x) = \int_a^x f(t)\, dt$$

If f is continuous on $[a, b]$, then $A(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Moreover $A'(x) = f(x)$. Equivalently, we can write

$$A'(x) = \frac{d}{dx}\left[\int_a^x f(t)\, dt \right] = f(x)$$

In words, we state that the area function $A(x)$ of f is an antiderivative of f on $[a, b]$.

Theorem 5.5. *p. 366* ***Fundamental Theorem of Calculus, Part 2***

Let $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined on the region $D = [a, b]$. If $F(x)$ is any antiderivative of f on the region $[a, b]$, then

$$\int_a^b f(x)\, dx = \int_a^b \left[\frac{d}{dx} [\, F(x)\,] \right] dx \qquad = F(b) - F(a)$$

In words, we state that the area function $A(x)$ of f is an antiderivative of f on $[a, b]$.

A good proof of this theorem is on pages 371 - 372. Please see that proof and recreate the arguments for yourself.

© Jeffrey A. Anderson