1. Let $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function on an interval $D = \{x \,:\, a \leq x \leq b\} \subseteq \mathbb{R}$. In this problem, we will derive the limit definition for the single integral of a function:

$$\int_a^b f(x)\ dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\ \Delta x_k$$

A. (6 points) Explain how to set up a general partition of D, how to choose a sample input value x_k^* from the kth subregion of the partition of the region D and to enumerate subregions from $k = 1, 2, ..., n$.

B. (6 points) Explain how to translate the Riemann sum $\sum_{k=1}^{n} f(x_k^*)\ \Delta x_k$ into the integral by taking a limit with respect to Δ where Δ is the maximum size of the subregions. With this in mind, please explain each symbol in integral notation.

Solution: Please see lesson notes and textbook for a detailed description of solution. Jeff is in the process of creating full typed solutions...

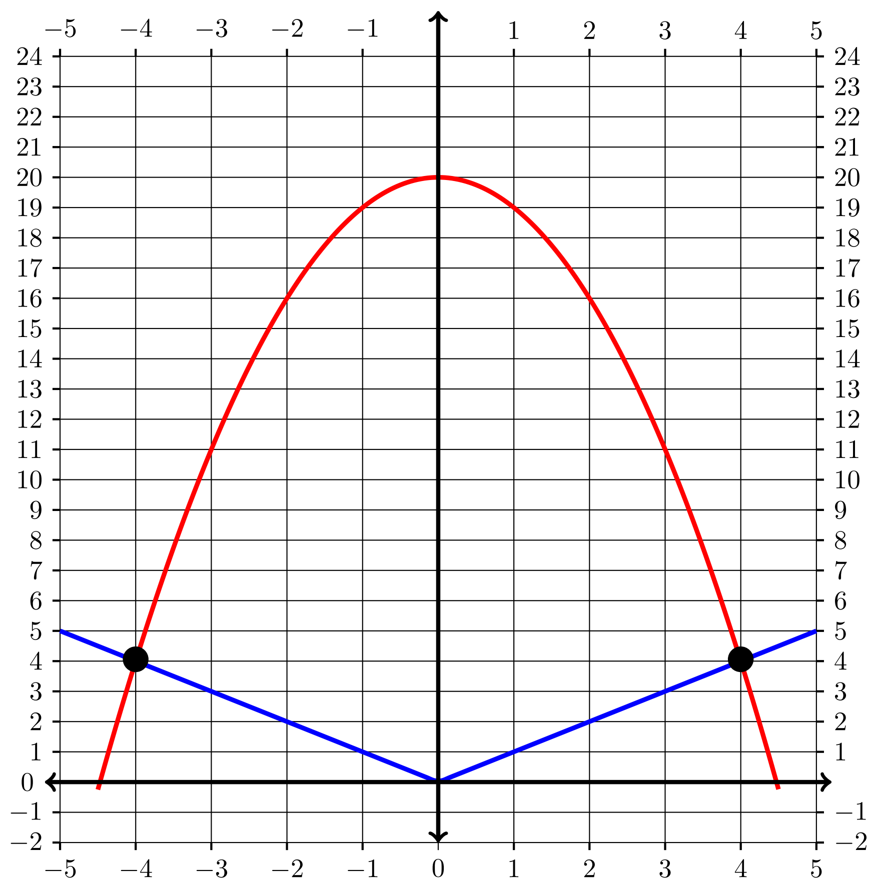
2. Consider the following integral

$$\iint_D f(x,y)\ dA$$

where the integrand $f(x,y) = x + y$ and $D \subseteq \mathbb{R}^2$ is the region bounded below by $y = |x|$ and above $y = 20 - x^2$

A. (6 points) Fill out the table below and sketch the region of integration

x	$y = \|x\|$	$y = 20 - x^2$
-5	5	-5
-4	4	4
-3	3	11
-2	2	16
-1	1	19
0	0	20
1	1	19
2	2	16
3	3	11
4	4	4
5	5	-5



Left P.o.I
$(-4, 4)$

Right P.o.I
$(4, 4)$

B. (6 points) Evaluate the integral: $\iint_D f(x,y)\; dA$ described in problem 2 above.

Solution: We will split our region into two y-simple subregions $D = D_1 \cup D_2$ where

$$D_1 = \{(x,y) \,:\, -4 \leq x \leq 0 \text{ and } -x \leq y \leq 20 - x^2\} \text{ and}$$

$$D_2 = \{(x,y) \,:\, 0 \leq x \leq 4 \text{ and } x \leq y \leq 20 - x^2\}$$

Once we've done this, we can write:

$$\iint_D f(x,y)\; dA = \iint_{D_1} f(x,y)\; dA + \iint_{D_2} f(x,y)\; dA$$

We can do each of these integrals separately. Let's start with the integral over the $y-$simple subregion D_1:

$$\iint_{D_1} f(x,y)\; dA = \int_{-4}^{0} \int_{-x}^{20-x^2} x + y \;\; dy \;\; dx$$

Let's deal with the inner integral first. To this end, consider:

$$\int_{-x}^{20-x^2} x + y \;\; dy = x \cdot y + \frac{y^2}{2}\Big|_{-x}^{20-x^2}$$

$$= x \cdot (20 - x^2 + x) + \frac{1}{2} \cdot \left((20 - x^2)^2 - (-x)^2\right)$$

$$= 20x + x^2 - x^3 + \frac{1}{2} \cdot \left(400 - 41x^2 + x^4\right)$$

$$= 200 + 20x - \frac{39}{2}x^2 - x^3 + \frac{x^4}{2}$$

Now, we can substitute this back into the outer integral:

$$\int_{-4}^{0} \left(200 + 20x - \frac{39}{2}x^2 - x^3 + \frac{x^4}{2} \right) dx = \frac{1,952}{5}$$

This integral is best done using a calculator. We now repeat this process for the second subregion D_2 to find:

$$\iint_{D_2} f(x,y)\; dA = \int_{0}^{4} \int_{x}^{20-x^2} x + y \;\; dy \;\; dx = \frac{8,096}{15}$$

Combining these two results together, we find our double integral on the given region:

$$\iint_D f(x,y)\; dA = \frac{13,952}{15}$$

3. (6 points) Evaluate the following integral:

$$\int_{-1}^{1} \int_{-1}^{2} \int_{0}^{1} 6\,x\,y\,z \;\; dy \; dx \; dz$$

Solution: Let's consider the triple integral given in the problem statement. We will begin by working with our inner-most integral:

$$\int_{0}^{1} 6\,x\,y\,z \;\; dy = 3 \cdot x \cdot z \cdot y^2 \;\Big|_{0}^{1} = 3 \cdot x \cdot z$$

Then, we can use this value to work on the middle integral:

$$\int_{-1}^{2} 3 \cdot x \cdot z \;\; dx = \frac{3}{2} \cdot z \;\cdot x^2 \;\Big|_{-1}^{2}$$

$$= \frac{3}{2} \cdot z \left(2^2 - (-1)^2\right) \qquad = \frac{15}{2} \cdot z$$

Finally, we end with the outer integral given by

$$\int_{-1}^{1} \frac{15}{2} \cdot z\,dz = \frac{15}{4} \cdot z^2 \;\Big|_{-1}^{1}$$

$$= \frac{15}{4} \cdot \left(1^2 - (-1)^2\right)$$

$$= 0.$$

4. Let $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function on a polar rectangle

$$D = \{(r, \theta) \,:\, a \leq r \leq b \text{ and } \alpha \leq \theta \leq \beta\}$$

where $z = f(r, \theta)$ is given in polar coordinates.

   A. (6 points) Explain how to set up the uniform discretization (a regular partition) of the polar rectangle D and derive the formula for the area of the kth sector of our partition as $\Delta A_k = r_k^* \cdot \Delta r \cdot \Delta\theta$

   B. (6 points) Now, explain why we define of the double integral of our function f on the polar rectangle as:

$$\iint_D f(r, \theta)\; dA = \int_\alpha^\beta \int_a^b f(r, \theta) \,\cdot r\; dr\;\; d\theta$$

   and explain why the integral has a factor of r in the differential form dA.

Solution: Please see lesson notes and textbook for a detailed description of solution. Jeff is in the process of creating full typed solutions...

5. (8 points) Use a double integral to find the volume of the solid bounded between the paraboloids:

$$z = x^2 + y^2 \qquad \text{and} \qquad z = 64 - 4x^2 - 4y^2.$$

Be sure to explain your reasoning and show your work.

Solution: We begin by setting up the desired double integral to find the volume of the solid describe in this problem:

$$\iint_D 64 - 5x^2 - 5y^2 \; dA$$

where the region $D = \left\{(x,y) : x^2 + y^2 \leq \frac{64}{5}\right\}$. We notice that encoding this problem using cartesian coordinates results in nasty arithmetic. Instead, let's transform into polar coordinates. To this end, we notice

$$D = \left\{(r,\theta) : 0 \leq \theta < 2\pi, 0 \leq r \leq \frac{8}{\sqrt{5}}\right\}$$

Then, we can write the following integral

$$\int_0^{2\pi} \int_0^{\frac{8}{\sqrt{5}}} \left(64 - 5r^2\right) \; r \cdot dr \;\; d\theta$$

Let's begin with the inner integral:

$$\int_0^{\frac{8}{\sqrt{5}}} \left(64 - 5r^2\right) \; r \cdot dr = 32r^2 - \frac{5}{4}r^4 \Big|_0^{\frac{8}{\sqrt{5}}}$$

$$= 32 \cdot \left(\frac{8}{\sqrt{5}}\right)^2 - \frac{5}{4} \cdot \left(\frac{8}{\sqrt{5}}\right)^4$$

$$= \frac{1024}{5}$$

Then, we use this result to evaluate the outer integral:

$$\int_0^{2\pi} \frac{1024}{5} \; d\theta = \frac{2048\,\pi}{5}$$

Optional Challenge Problem

6. (5 points) Show that

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}.$$

Solution: If we assume that

$$I = \int_{-\infty}^{\infty} e^{-x^2}\ dx,$$

then we can state that

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\ dx \right) \cdot \left(\int_{-\infty}^{\infty} e^{-y^2}\ dy \right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\ dx\ dy$$

$$= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-(r^2)} \cdot r\ dr\ d\theta = \int_{0}^{2\pi} A(\theta)\ d\theta = \int_{0}^{2\pi} \frac{1}{2}\ d\theta = \pi$$

Then, by taking the square root of both sides, we see that $I = \sqrt{\pi}$, which is exactly what we wanted to show. In the last step above, we transformed the double integral in rectangular coordinates into a single iterated integral with respect to θ, where we set

$$A(\theta) = \int_{0}^{\infty} e^{-(r^2)} \cdot r\ dr \qquad\qquad \text{let } u = r^2 \longrightarrow \frac{1}{2}\, du = r\, dr$$

$$= \lim_{t \to \infty} \int_{0}^{t} \frac{1}{2} \cdot e^{-u}\ du$$

$$= \lim_{t \to \infty} -\frac{1}{2} \cdot e^{-u} \Big|_{0}^{t}$$

$$= -\frac{1}{2} \lim_{t \to \infty} \left(\frac{1}{e^t} - \frac{1}{e^0} \right) = \frac{1}{2}$$