True/False

(15 points: 3 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

1. T **F** If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

2. T **F** The series $\sum_{n=1}^{\infty} 3ne^{-n^2}$ diverges

3. **T** F If f has a local minimum at point $(a, b) \in \mathbb{R}^2$, then

$$D_{\mathbf{u}f(a,b)} = 0$$

for any unit vector $\mathbf{u} \in \mathbb{R}^2$.

4. **T** F If $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

5. T **F** Suppose $f : \mathbb{R}^3 \to \mathbb{R}$. If $\nabla f = 0$ at a point $\mathbf{x} \in \mathbb{R}^3$, then f has a local extreme value at point $\mathbf{x}$.

Multiple Choice

(45 points: 3 points each) For the problems below, circle the correct response for each question. After you've chosen your answer, mark your answer on your Scantron form.

6. Let $z = \sin(x \cdot y)$ and let $x = x(t)$ and $y = y(t)$ be functions of t. Suppose

$$x(1) = 0, \qquad y(1) = 1, \qquad x'(1) = 2, \qquad y'(1) = 3.$$

Find $\dfrac{dz}{dt}$ when $t = 1$.

A. 1 **B.** 2 C. 3 D. 4 E. 5

7. The series $\sum_{n=0}^{\infty} r^n$ converges if and only if:

A. $-1 < r < 1$ B. $-1 \le r \le 1$ C. $-1 \le r < 1$ D. $-1 < r \le 1$ E. $r < 1$

8. Find the direction of maximum increase of the function $f(x, y, z) = x\,e^{-y} + 3\,z$ at the point $(1, 0, 4)$.

A. $\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$ **B.** $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ C. $\begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$ D. $\begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$ E. $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

9. Which of the following series converge?

1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ 2) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{\ln(n)}$ 3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

A. 1 B. 2 **C.** 3 D. 2, 3 E. None

10. Find the limit of the sequence $a_n = 2 + \left(-\frac{4}{5}\right)^n$:

A. 2 B. $\frac{6}{5}$ C. $-\frac{4}{5}$ D. -2 E. $\frac{4}{5}$

11. Find the shortest distance from the origin to the surface $z^2 = 2xy + 2$

A. $\frac{1}{\sqrt{2}}$ **B.** $\sqrt{2}$ C. $\frac{1}{2}$ D. 2 E. 1

12. The series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges if and only if

A. $1 < \alpha$ B. $\alpha < 1$ C. $-1 < \alpha < 1$ D. $\alpha \geq 1$ E. $-1 < \alpha$

13. Find the minimum value of the function $f(x, y) = x\,y$ subject to the constraint that $x^2 + y^2 = 2$:

A. 1 B. 2 **C.** -1 D. $\frac{3}{2}$ E. $-\frac{3}{2}$

14. Find the values of x for which the series $\sum_{n=1}^{\infty} (x-1)^n$:

A. $-2 < x < 0$ B. $0 < x \leq 2$ **C.** $0 < x < 2$ D. $0 \leq x \leq 2$ E. $-2 \leq x < 0$

15. Find the directional derivative of the function

$$f(x, y) = y^2 \cdot \ln(x)$$

at the point $(1, 2)$ in the direction of the vector $(3, 4) = 3\mathbf{i} + 4\mathbf{j}$:

A. $\frac{5}{16}$ B. 12 C. $\frac{5}{12}$ D. $\frac{16}{5}$ **E.** $\frac{12}{5}$

16. Find an equation of the tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 4$ at the point $(4, 1, 1)$.

A. $2x + y - z = 1$
B. $x + 2y + 2z = 8$
C. $x - 2y + 4z = 0$
D. $x + y + z = 6$
E. $2x + y + z = 10$

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17. Which of the following series converge?

1) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^{3n}}$ 2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$ 3) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n^3+2}}$

A. None B. 1 **C.** 2 D. 3 E. 2, 3

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18. Determine how many critical points the function $f(x, y) = x\,y - x^2\,y - x\,y^2$ has:

A. 1 B. 2 C. 3 **D.** 4 E. 5

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19. How many terms of the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} n^{-2}$$

must we add in order to be sure that the partial sum s_n is within 0.0001 of the sum s.

A. 10 B. 300 C. 30 **D.** 100 E. 1000

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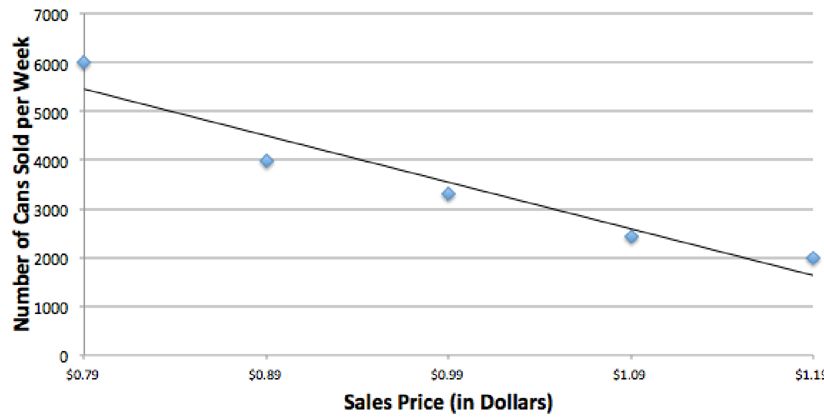
20. Let $f(x, y) = \dfrac{x}{y} + \dfrac{y}{x}$. Find the gradient vector ∇f:

A. $\begin{bmatrix} 2y \\ 2x \end{bmatrix}$ B. $\begin{bmatrix} x \\ y \end{bmatrix}$ **C.** $\begin{bmatrix} \dfrac{1}{y} - \dfrac{y}{x^2} \\ \dfrac{1}{x} - \dfrac{x}{y^2} \end{bmatrix}$ D. $\begin{bmatrix} -y/x^2 \\ -x/y^2 \end{bmatrix}$ E. $\begin{bmatrix} y \\ x \end{bmatrix}$

Free Response

21. (10 points) A company test-markets a new canned energy drink made of all natural ingredients in 5 cities of equal size on the West Coast of the US. The selling price (in dollars) and the number of drinks sold per week is each of the cities is listed as follows

City	Price	Sales/Week
1	0.79	6000
2	0.89	3980
3	0.99	3300
4	1.09	2440
5	1.19	1990



This data is plotted in the figure next to the table above. Although the data do not exactly lie on a straight line, we can create a linear model to fit this data.

A. Set up the least squares problem to fit this data to a linear model

$$S(p) = c_1 + c_2 p$$

where S is the sales per week and p is the price.

Solution: Recall that the least squares problem is designed to fit data collected during an experiment to a particular mathematical model. In this case, we are told that our company collected five data points $\{(p_i, s_i)\}_{i=1}^{5}$, where

$$p_i = \text{the price per energy drink sold in city } i \text{ for } i = 1, 2, ..., 5$$
$$s_i = \text{the number of cans of the energy drink sold in city } i \text{ for } i = 1, 2, ..., 5$$

We notice that the model appears to fit a linear model $S(p) = c_1 + c_2 p$ for unknown parameters $c_1, c_2 \in \mathbb{R}$. This model can be used to predict the number of cans sold in city i based on the selling price p_i as follows:

$$S(p_i) = c_1 + c_2 \cdot p_i.$$

The difference between the observed data and the model prediction is known as the model error in the ith term, given by:

$$e_i = (S(p_i) - s_i) = (c_1 + c_2 \cdot p_i - s_i).$$

To create the model of best fit for unknown parameters $c_1, c_2 \in \mathbb{R}$, we want to minimize the sum of the squared error terms:

$$f(c_1, c_2) = \sum_{i=1}^{5} e_i^2$$

$$= \sum_{i=1}^{5} (c_1 + c_2 \cdot p_i - s_i)^2$$

$$= (c_1 + 0.79 \cdot c_2 - 6000)^2 + (c_1 + 0.89 \cdot c_2 - 3980)^2$$

$$+ (c_1 + 0.99 \cdot c_2 - 3300)^2 + (c_1 + 1.09 \cdot c_2 - 2440)^2 + (c_1 + 1.19 \cdot c_2 - 1990)^2$$

Thus, the least squares problem is to minimize the function $f(c_1, c_2)$.

B. Explain how you would use multivariable calculus to find the line of best fit.

Solution: We apply multivariable calculus to solve this problem by recalling the second derivative test for the multivariable function $f(c_1, c_2)$. In particular, we know f has a local minimum if and only if

A. $\nabla f = \mathbf{0}$

B. $\dfrac{\partial f}{\partial c_1} \cdot \dfrac{\partial^2 f}{\partial c_2} - \left(\dfrac{\partial f}{\partial c_1 \partial c_2} \right)^2 < 0$ with $\dfrac{\partial^2 f}{\partial c_1^2} > 0$.

Thus, to find the local minimum of f using multivariable calculus, we need to find the critical points of this function and apply the second derivative test for multivariable function appropriately.

Remark (preview of coming attractions): There are two drawbacks of this method worth mentioning:

I. The method of minimizing the square of the modeled error is algebraically intensive. It requires us to expand the multivariable function $f(c_1, c_2)$ into quadratic terms in c_1 and c_2. Further to find the zeros of this polynomial requires non-linear methods.

II. Although multivariable calculus can be used to verify that the critical point where $\nabla f = \mathbf{0}$ is a local minimum, there is theoretical result that can conclude that this point will also be a global minimum. Thus, without further analysis of the function f, this method will not always guarantee a unique absolute minimum error term.

In Math 2B (Linear Algebra), we will revisit this problem using least squares techniques to improve the methods we discussed in this class.

22. (10 points)

A. Find the value of $\sum_{n=2}^{\infty} \frac{3^n + 5^n}{15^n}$. Show your work.

Solution:

$$\sum_{n=2}^{\infty} \frac{3^n + 5^n}{15^n} = \sum_{n=2}^{\infty} \frac{3^n}{15^n} + \frac{5^n}{15^n}$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{5}\right)^n + \left(\frac{1}{3}\right)^n$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{n+1} + \left(\frac{1}{3}\right)^{n+1}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^2 \cdot \left(\frac{1}{5}\right)^{n-1} + \left(\frac{1}{3}\right)^2 \cdot \left(\frac{1}{3}\right)^{n-1}$$

$$= \frac{1}{25} \cdot \frac{1}{1-\frac{1}{5}} + \frac{1}{9} \cdot \frac{1}{1-\frac{1}{3}} = \boxed{\frac{13}{60}}$$

B. Show that series $\sum_{n=2}^{\infty} \frac{1}{n \cdot (\ln(n))^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Solution: In this problem, we are asked to find values of p such that our given series converges. To this end, let

$$f(x) = \frac{1}{x \cdot (\ln(x))^p}.$$

For $x \geq 2$, we confirm that $f(x)$ is positive, decreasing and continuous using methods from Math 1A. Thus, by the integral test for convergence, we know that our given series and the integral

$$\int_2^{\infty} f(x)dx$$

have identical convergence behavior (the either both converge or they both diverge). With this in mind, consider the two cases:

I. Case $p = 1$:

$$\int_2^\infty f(x)dx = \int_2^\infty \frac{1}{x \cdot \ln(x)} dx$$

$$= \int_{\ln(2)}^\infty \frac{1}{u} du \qquad \text{Let } u = \ln(x) \Longrightarrow du = \frac{1}{x} dx \text{ and}$$

$$u(2) = \ln(2) \text{ while } u(\infty) = \infty$$

$$= \lim_{t \to \infty} \int_{\ln(2)}^t \frac{1}{u} du$$

$$= \lim_{t \to \infty} \ln(u)\Big|_{\ln(2)}^t$$

$$= \left(\lim_{t \to \infty} \ln(t)\right) - \ln(2) = \infty.$$

Thus, for $p = 1$ our integral diverges. By the integral test for series, we conclude that the given series also diverges for $p = 1$.

II. Case $p \neq 1$:

$$\int_2^\infty f(x)dx = \int_2^\infty \frac{1}{x \cdot (\ln(x))^p} dx$$

$$= \int_{\ln(2)}^\infty \frac{1}{u^p} du \qquad \text{Let } u = \ln(x) \Longrightarrow du = \frac{1}{x} dx \text{ and}$$

$$u(2) = \ln(2) \text{ while } u(\infty) = \infty$$

$$= \lim_{t \to \infty} \int_{\ln(2)}^t u^{-p} du$$

$$= \lim_{t \to \infty} \frac{u^{1-p}}{1-p}\Big|_{\ln(2)}^t$$

$$= \left(\lim_{t \to \infty} \frac{t^{1-p}}{1-p}\right) + c \text{ where } c = -\frac{(\ln(2))^{1-p}}{1-p}.$$

Thus we see that the integral converges if $p > 1$ and diverges for all $p \leq 1$. This is what was to be shown.

23. (10 points) Let $f : \mathbb{R}^2 \to R$ be a differentiable, multivariable function. Let $\mathbf{u}$ be a unit vector.

A. Derive the dot product formula for the limit definition of the directional derivative $D_{\mathbf{u}}f$.

Solution: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a multivariable function with two input variables. Suppose $\mathbf{u} = a\mathbf{i} + b\mathbf{j} = (a, b)$ be a unit vector. To find the directional derivative of f at a point $\mathbf{x}_0 = (x_0, y_0) \in \mathbb{R}^2$ in the direction of $\mathbf{u}$, let $\mathbf{x} = \mathbf{x}_0 + t\,\mathbf{u}$ us consider the following limit:

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|_2} = \lim_{t\to 0} \frac{f(x_0 + at, y_0 + bt) - f(x_0, y_0)}{t} = g'(0)$$

where we introduce the auxiliary function

$$g(t) = f(x_0 + a\,t, y_0 + b\,t).$$

For reference, the conversion from the first limit to the second follows from function evaluation at the vectors $\mathbf{x}$ and $\mathbf{x}_0$ along with the calculation:

$$\|\mathbf{x} - \mathbf{x}_0\|_2 = \|t\mathbf{u}\|_2 = |t|.$$

We can now use the auxiliary function $g(t)$ combined with the chain rule for multivariable functions to find an equivalent representation for $g'(t)$:

$$g'(t) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= f_x(x_0 + a\,t, y_0 + b\,t)a + f_y(x_0 + a\,t, y_0 + b\,t)b$$

Substituting the value $t = 0$ into this equation leads to

$$D_{\mathbf{u}}f(\mathbf{x}_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \nabla f \cdot \mathbf{u}.$$

B. Prove that the gradient vector is in the direction of maximum increase.

Solution: From Part A above, we know can use the dot product formula for the directional derivative of f in the direction of $\mathbf{u}$ to find

$$\begin{aligned} D_{\mathbf{u}}f &= \|\nabla f \cdot \mathbf{u}\|_2 \\ &= \|\nabla f\|_2 \ \|\mathbf{u}\|_2 \ \cos(\theta) \\ &= \|\nabla f\|_2 \ \cos(\theta) \\ &\le \|\nabla f\|_2 \end{aligned}$$

We used the cosine formula for the dot product to get this series of equalities where θ is the angle between the gradient vector and the vector $\mathbf{u}$. Thus we see that the directional derivative is maximized when $\cos(\theta) = 1$, which happens when $\theta = 90°$. In other words, the gradient points in the direction of maximal increase.

Challenge Problem

24. (Optional, Extra Credit, Challenge Problem)

Suppose that a sequence $\{a_n\}_{n=1}^{\infty}$ satisfies

$$0 < a_n \leq a_{2n} + a_{2n+1}$$

for all $n \geq 1$. Prove that the series $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Solution: Let us begin this problem by considering the significance of the stated property that $0 < a_n \leq a_{2n} + a_{2n+1}$ for our sequence $\{a_n\}_{n=1}^{\infty}$. Consider

$$\begin{aligned} a_1 &\leq a_2 + a_3 \\ a_2 + a_3 &\leq a_4 + a_5 + a_6 + a_7 \\ a_4 + a_5 + a_6 + a_7 &\leq a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} \end{aligned}$$

Notice we can find a pattern in the indices for these inequalities:

$$\begin{aligned} 1 &\leq 2 + 3 \\ 2 + 3 &\leq 4 + 5 + 6 + 7 \\ 4 + 5 + 6 + 7 &\leq 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 \end{aligned}$$

Let's use this pattern to generalize to the kth inequality. Set

$$p_k = \sum_{j=2^{k-1}}^{2^k - 1} a_j$$

and notice we can write $p_k \leq p_{k+1}$. Thus, we must have

$$\sum_{n=1}^{2^N - 1} a_j = \sum_{k=1}^{N} p_k \geq N p_1 = N a_1.$$

As $N \to \infty$ we see the partial sums are unbounded since $a_1 > 0$ and thus the original series must diverge as claimed.