True/False

(10 points: 2 points each) For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

1. **(T)** F $f_y(a,b) = \lim_{y\to b} \frac{f(a,y) - f(a,b)}{y-b}$.

2. **(T)** F For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$.

3. T **(F)** The set of points $\{(x,y,z) : x^2 + y^2 = 1\}$ is a circle.

4. T **(F)** If $f(x,y) \to L$ as $(x,y) \to (a,b)$ along every straight line through (a,b),, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

5. **(T)** F If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, then $|\mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}\|_2 \, \|\mathbf{y}\|_2$

Multiple Choice

(60 points: 4 points each) For the problems below, circle the correct response for each question. After you've chosen your answer, mark your answer on your Scantron form.

6. Which of the following is a unit vector point in the direction of vector $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

   A. $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ **B.** $\frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ C. $\frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ D. $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ E. $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$

7. Consider the vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}.$$

   Which of the following vectors gives $\mathbf{x} \cdot \mathbf{y}$?

   A. 14 B. 10 C. $-12\mathbf{k}$ D. -10 **E.** -14

8. Find values of $b \in \mathbb{R}$ such that the vectors $\begin{bmatrix} -11 \\ b \\ 2 \end{bmatrix}$ and $\begin{bmatrix} b \\ b^2 \\ b \end{bmatrix}$ are orthogonal.

   A. $0, 11, -3$ B. $0, -11, 2$ C. $0, 2, -2$ **D.** $0, 3, -3$ E. $0, 11, 2$

9. Given $\mathbf{x} = (4, 0)$ and $\mathbf{y} = (5, 2)$, which of the following is the projection of vector $\mathbf{x}$ onto the vector $\mathbf{y}$?

   A. $(5, 0)$ B. $\left(\frac{40}{27}, \frac{16}{27}\right)$ **C.** $\left(\frac{100}{29}, \frac{40}{29}\right)$ D. $(4, 2)$ E. $\left(\frac{100}{\sqrt{29}}, \frac{40}{\sqrt{29}}\right)$

10. Consider the vectors

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = 2\mathbf{i} + 3\mathbf{k}, \qquad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \mathbf{j} - \mathbf{k}.$$

   Which of the following vectors gives $\mathbf{x} \times \mathbf{y}$?

   A. $-3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ B. $-3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ C. -3 D. $-3\mathbf{k}$ E. $3\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$

11. Find an equation of the line through the point $(1, 2, 3)$ and parallel to the plane $x - y + z = 100$:

A. $x - y + z - 2 = 0$

B. $x - y + z + 2 = 0$

C. $x + 2y + 3z = 100$

D. $x - 1 = 2 - y = z - 3$

E. $x - 1 = \dfrac{y+1}{2} = \dfrac{z-1}{3}$

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12. Given $f(x, y) = \sqrt{x^2 + y^2}$, find f_{xx} :

A. $\dfrac{xy}{\left(x^2+y^2\right)^{1/2}}$

B. $\dfrac{x^2}{\left(x^2+y^2\right)^{3/2}}$

C. $\dfrac{y^2}{\left(x^2+y^2\right)^{3/2}}$

D. $\dfrac{y}{\left(x^2+y^2\right)^{1/2}}$

E. $\dfrac{x}{\left(x^2+y^2\right)^{1/2}}$

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13. Find an equation for the line through the point $(3, -1, 2)$ and perpendicular to the plane $2x - y + z + 10 = 0$.

A. $\dfrac{x-3}{2} = \dfrac{y+1}{-1} = z - 2$

B. $\dfrac{x+2}{2} = \dfrac{y-1}{-1} = z - 2$

C. $\dfrac{x-2}{3} = \dfrac{y+1}{-1} = \dfrac{x-2}{2}$

D. $3x - y + 2z + 10 = 0$

E. $3x - 2y + z + 10 = 0$

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14. Find the area of the triangle with vertices at the points $(0, 0, 0)$, $(1, 0, -1)$ and $(1, -1, 2)$.

A. $\sqrt{11}$

B. $\sqrt{6}$

C. $\dfrac{\sqrt{6}}{2}$

D. $\dfrac{\sqrt{11}}{2}$

E. 1

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15. The equation of the sphere with center $(4, -1, 3)$ and radius $\sqrt{5}$ is

A. $(x-4)^2 + (y+1)^2 + (z-3)^2 = 5$

B. $(x-4)^2 + (y+1)^2 + (z-3)^2 = \sqrt{5}$

C. $(x-4)^2 + (y+1)^2 + (z-3)^2 = 25$

D. $(x+4)^2 + (y-1)^2 + (z+3)^2 = 5$

E. $(x-4)^2 + (y-1)^2 + (z-3)^2 = 5$

16. Find the distance between the point $(-1, -1, -1)$ and the plane $x + 2y + 2z - 1 = 0$

A. 0 B. 6 C. -2 D. -6 **E. 2**

17. Let $f(x, y) = e^{\sin(x)} + x^5 y + \ln\left(1 + y^2\right)$. Find f_{yx} :

A. $5x^4$ B. $\frac{2y}{1+y^2}$ C. $20x^3y$ D. $e^{\sin(x)}\cos(x)$ E. $e^{\sin(x)}\cos(x) + x^5 + \frac{2y}{1+y^2}$

18. Find the limit $\lim_{(x,y)\to(0,0)} \frac{2x^4y^2}{x^4 + 3y^4}$

A. 2 B. $\frac{2}{3}$ **C.** 0 D. $\frac{1}{2}$ E. Does NOT exist

19. Find the parametric equations of the intersection of the planes $x - z = 0$ and $x - y + 2z + 3 = 0$

A. The line given by $x(t) = -2 - t, y(t) = 1 - 3t$ and $z(t) = -t$.
B. The line given by $x(t) = 1 + t, y(t) = 6 - t$ and $z(t) = 1 + 2t$.
C. The plane $3x + 3y - 3z + 3 = 0$
D. The line given by $x(t) = -t, y(t) = 3 - 3t$ and $z(t) = -t$.
E. The line given by $x(t) = 1 + t, y(t) = 6$ and $z(t) = 1 - t$.

20. Given $\mathbf{x} = (2, 0, 1)$ and $\mathbf{v} = (4, 1, 2)$, what is the area of the parallelogram formed by the vectors $\mathbf{x}$ and $\mathbf{y}$?

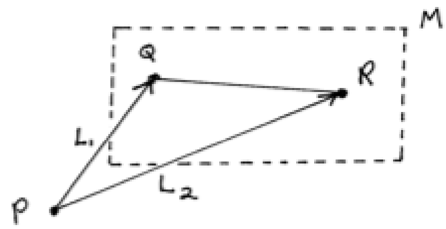
A. $2\sqrt{5}$ **B.** $\sqrt{5}$ C. $2\sqrt{3}$ D. $3\sqrt{2}$ E. $4\sqrt{2}$

Free Response

21. Consider the lines

$$\mathbf{L}_1(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 1-t \end{bmatrix} \qquad \text{and} \qquad \mathbf{L}_2(s) = \begin{bmatrix} x(s) \\ y(s) \\ z(s) \end{bmatrix} = \begin{bmatrix} 1-s \\ 2+s \\ -2s \end{bmatrix}$$

and the plane M given by equation $10x - 2y + 3z = 0$. The line $\mathbf{L}_1(t)$ intersects plane M at point Q. The line $\mathbf{L}_2(s)$ intersects plane M at point R. Lines $\mathbf{L}_1(t)$ and $\mathbf{L}_2(s)$ intersect at point P. Compute the area of the triangle PQR. The diagram below may help you visualize this problem.



Solution: Let's begin this problem by finding the coordinates of points P, Q and R.

i. Coordinates of Point P:

Since point P represents the intersection of lines $\mathbf{L}_1(t)$ and $\mathbf{L}_2(s)$, we want to set the parametric equations for the lines equal and solve for parameter values t and s to produce a single point P on both lines. We begin by forming the system of three equations in two unknowns given by

$$\mathbf{L}_1(t) = \mathbf{L}_2(s) \qquad \Longrightarrow \qquad \begin{bmatrix} t \\ 2t \\ 1-t \end{bmatrix} = \begin{bmatrix} 1-s \\ 2+s \\ -2s \end{bmatrix}.$$

Since all three of these equations must hold true and we only have two unknowns, let's solve the first two equations for t and s by elimination. Since $x(t) = x(s)$ yields

$$t = 1 - s.$$

we can substitute this into the equation that results from setting $y(t) = y(s)$ to find

$$2 \cdot (1 - s) = 2 + s$$

We find that at point P we must have that $s = 0$ and $t = 1$. Substituting these values back into the parametric equations for our lines, we find the coordinates of point P to be

$$P(1, 2, 0)$$

ii. Coordinates of Point Q:

Since point Q represents the intersection of line $\mathbf{L}_1(t)$ and the plane M, we can substitute the parametric equations that define the $x-, y-$ and z-coordinates of line $\mathbf{L}_1(t)$ into the equation for the plane to find the value of our parameter t at this point of intersection. This calculation goes as follows:

$$
\begin{aligned}
10x(t) - 2y(t) + 3z(t) = 0 \quad &\Longrightarrow \quad 10 \cdot t - 2 \cdot 2t + 3 \cdot (1 - t) = 0 \\
&\Longrightarrow \quad 3t + 3 = 0 \\
&\Longrightarrow \quad t = -1
\end{aligned}
$$

Now that we have the parameter value of t at point Q, we can substitute this value back into our equation for the line $\mathbf{L}_1(t)$ to find the coordinates of point Q

$$\mathbf{L}_1(-1) = Q(-1, -2, 2).$$

iii. Coordinates of Point R:

Since point R represents the intersection of line $\mathbf{L}_2(s)$ and the plane M, we can substitute the parametric equations that define the $x-, y-$ and z-coordinates of line $\mathbf{L}_2(s)$ into the equation for the plane to find the value of our parameter s at this point of intersection. This calculation goes as follows:

$$
\begin{aligned}
10x(s) - 2y(s) + 3z(s) = 0 \quad &\Longrightarrow \quad 10 \cdot (1 - s) - 2 \cdot (2 + s) + 3 \cdot (-2s) = 0 \\
&\Longrightarrow \quad 6 - 18s = 0 \\
&\Longrightarrow \quad s = \frac{1}{3}
\end{aligned}
$$

Now that we have the parameter value of s at point R, we can substitute this value back into our equation for the line $\mathbf{L}_2(s)$ to find the coordinates of point R

$$\mathbf{L}_2(1/3) = R\left(\frac{2}{3}, \frac{7}{3}, -\frac{2}{3}\right).$$

Now that we have the coordinates of points P, Q and R, let's focus on the computing the desired area of our triangle. Create the vectors

$$\mathbf{x} = \overrightarrow{PQ} = \langle -2, -4, 2 \rangle \qquad \text{and} \qquad \mathbf{y} = \overrightarrow{QR} = \left\langle \frac{5}{3}, \frac{13}{3}, -\frac{8}{3} \right\rangle$$

that connect P to Q and Q to R, respectively. We know that the area of our triangle will be

$$\frac{\| \, \mathbf{x} \times \mathbf{y} \, \|_2}{2} = \frac{\| \, \langle 2, -2, -2 \rangle \, \|_2}{2} = \boxed{\sqrt{3}}$$

22. Let $z = z(x, y)$ be defined implicitly by equation

$$x^2 + y^2 - z^2 = 3xyz$$

Compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at point $(3, 1, 1)$.

Solution: Let's check that point $(x, y, z) = (3, 1, 1)$ is on the surface described by the given equation:

$$x^2 + y^2 - z^2 = (3)^2 + 1^2 - 1^2 = 9 = 3 \cdot 3 \cdot 1 \cdot 1 = 3xyz$$

Next, we notice that the given equation gives an implicit relationship between the variables x, y and z. With a little thought, we see that it is algebraically very messy to try to solve for z explicitly in terms of x and y. Instead, let's apply the implicit differentiation technique we discussed in Lesson 10. To do so, we need to assume $z = z(x, y)$ is a differentiable function of x and y. Now, to find $\frac{\partial z}{\partial x}$, we apply the partial derivative with respect to x to both sides of our equation

$$\frac{\partial}{\partial x}\left[\ x^2 + y^2 - z^2\ \right] = \frac{\partial}{\partial x}\left[\ 3xyz\ \right]$$

$$\Longrightarrow \qquad \frac{\partial}{\partial x}\left[\ x^2\ \right] + \frac{\partial}{\partial x}\left[\ y^2\ \right] - \frac{\partial}{\partial x}\left[\ z^2\ \right] = 3y\,\frac{\partial}{\partial x}\left[\ x \cdot z\ \right]$$

$$\Longrightarrow \qquad 2x - 2z\,\frac{\partial z}{\partial x} = 3yz + 3xy\,\frac{\partial z}{\partial x}$$

The last line in this inequality comes from treating z as a function of x and applying the proper rules of differentiation including Chain Rule. Now, we can isolate the partial derivative of z with respect to x on one side of the equation to find

$$\left.\frac{\partial z}{\partial y}\right|_{(3,1,1)} = \left.\frac{2x - 3yz}{3xy + 2z}\right|_{(3,1,1)} = \boxed{\frac{3}{11}}$$

We find $\frac{\partial z}{\partial y}$ in a similar manner. We begin by taking the partial derivative with respect to y of both sides of our original equation and applying the same techniques as above

$$\frac{\partial}{\partial y}\left[\ x^2 + y^2 - z^2\ \right] = \frac{\partial}{\partial y}\left[\ 3xyz\ \right]$$

$$\Longrightarrow \qquad \frac{\partial}{\partial y}\left[\ x^2\ \right] + \frac{\partial}{\partial y}\left[\ y^2\ \right] - \frac{\partial}{\partial y}\left[\ z^2\ \right] = 3x\,\frac{\partial}{\partial y}\left[\ y \cdot z\ \right]$$

$$\Longrightarrow \qquad 2x - 2z\,\frac{\partial z}{\partial y} = 3yz + 3xy\,\frac{\partial z}{\partial y}$$

Again, we isolate the partial derivative of z with respect to y on one side of the equation to find

$$\left.\frac{\partial z}{\partial y}\right|_{(3,1,1)} = \left.\frac{2y - 3xz}{3xy + 2z}\right|_{(3,1,1)} = \boxed{-\frac{7}{11}}$$

23. A. Use a scalar projection to show that the distance from point $P(x_1, y_1)$ to line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

Draw a diagram and explain your reasoning in detail using full sentences.

Solution:

Case 1: If $b = 0$, then we have a vertical line given by $x = \frac{-c}{a}$. the distance from the point $P(x_1, y_1)$ to this vertical line will be the length of the horizontal line segment that connects point P to this line. In other words, this distance will be

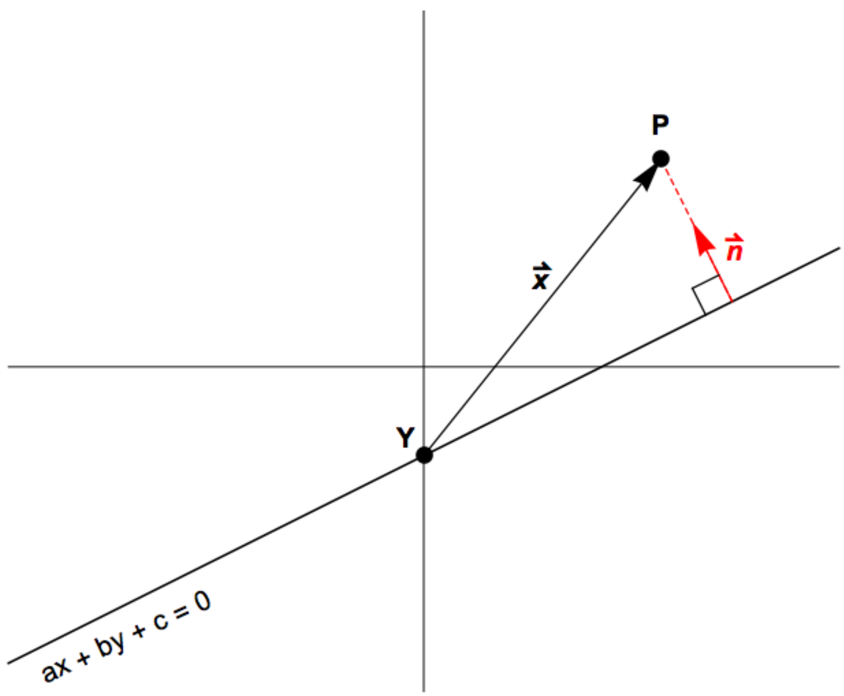
$$\left| x_1 - \frac{-c}{a} \right| = \left| \frac{ax_1 + c}{a} \right| = \frac{|ax_1 + c|}{\sqrt{a^2}}$$

The last equality comes from the fact that $|a| = \sqrt{a^2}$. But, this is exactly the formula we are asked to prove if $b = 0$.

Case 2: Assume $b \neq 0$. Consider the given equation for our line $ax + by + c = 0$. We can transform this equation into slope-intercept form using our knowledge of algebra. To this end, consider

$$ax + by = -c \qquad \Longrightarrow \qquad y = \frac{-a}{b}x + \frac{-c}{b}$$

The slope of this line is $m = \frac{-a}{b}$ and the $y-$ intercept is at the point $Y\left(0, \frac{-c}{b}\right)$. Let's graph this line on the cartesian plane, below, along with the point P.



By our discussion in Lesson 7, we know that the normal vector to this line is given by $\mathbf{n} = \langle a, b \rangle$. We construct the vector $\mathbf{x} = \overrightarrow{YP}$ that starts at the $y-$intercept Y and points toward the point P. This vector has coordinates

$$\mathbf{x} = \left\langle x_1, y_1 + \left(\frac{c}{b}\right) \right\rangle$$

To find the distance from point P to the given line $ax + by + c = 0$, we calculate the scalar component of $\mathbf{x}$ in the direction of $\mathbf{n}$. This is given by the equation

$$\text{Scal}_{\mathbf{n}}(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{n}}{\| \mathbf{n} \|_2} = \frac{\left\langle x_1, y_1 + \left(\frac{c}{b}\right) \right\rangle \cdot \langle a, b \rangle}{\sqrt{a^2 + b^2}} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

This is the exact formula we wanted to derive.

B. Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

Solution: Now we apply the formula we derived in Part A. to the line $3x - 4y + 5 = 0$ and the point $P(-2, 3)$. Notice that we have

$$a = 3, \qquad b = -4, \qquad c = 5, \qquad x_1 = -2, \qquad y_1 = 3$$

With this, we see that the distance from the point P to the given line is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} = \boxed{\frac{13}{5}}$$