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lesson 9 Suggested problem (Part I).

1. Let $f: D \subseteq \mathbb{R} \to \mathbb{R}$ be single variable, real-valued function that is differentiable on domain D. Derive the limit definition of the ordinary derivative of f at input a, given by:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Solution:

we construct the derivative operation with the idea of the secant line:

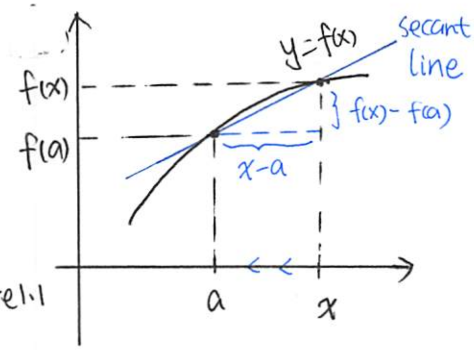

Figure 1.1

let $x \to a$

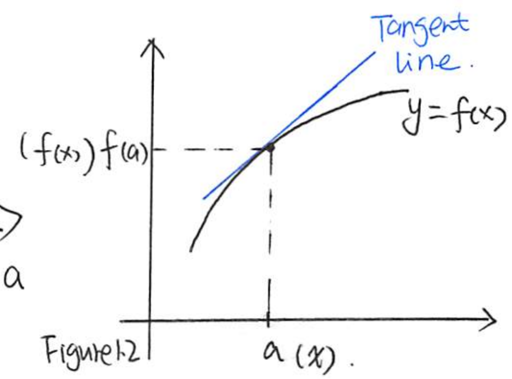

Figure 1.2

we can measure the slope of the secant line thru $(a, f(a))$ and $(x, f(x))$ by slope formula:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{f(x) - f(a)}{x - a}$$ Figure 1.1

To find the slope at point $(a, f(a))$,

we force the input x to a,

then change of input $(x-a)$ approaches to zero,

& change of output $(f(x)-f(a))$ goes to some number,

Let's say:

$\lim_{x \to a} (x-a)$ (I)

$\lim_{x \to a} f(x)-f(a)$ (II).

the ratio: $\frac{(\text{II})}{(\text{I})}$ is the slope of the curve at $(a, f(a))$.

slope of this tangent line at $x=a$:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$ Figure 1.2

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2. Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a two-variable, real valued function that is differentiable on domain D. Discuss the five step process for constructing the partial derivative of f with respect to x.

A. derive limit definition of partial derivative in slope notation

$$f_x(a,b) = \lim_{x \to a} \frac{f(x,b) - f(a,b)}{x - a}$$

Figure 2.1

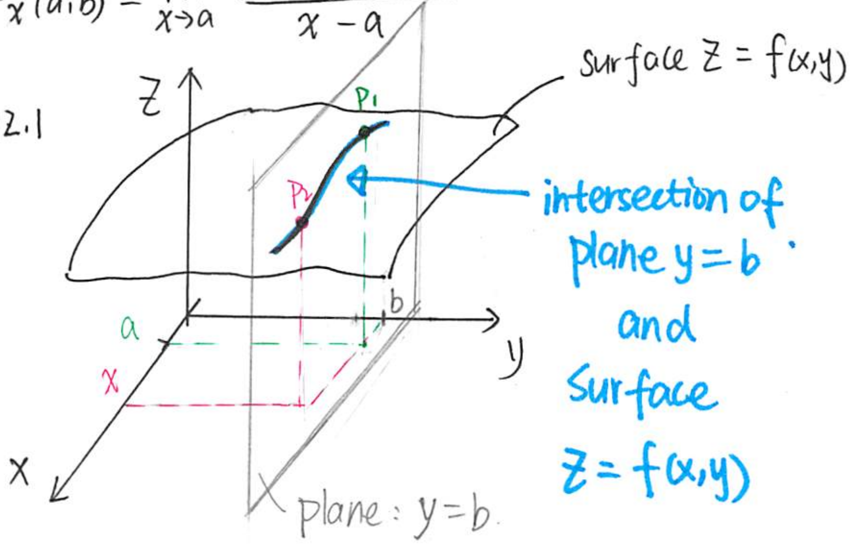
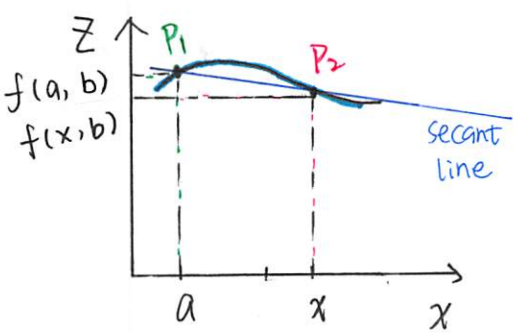


Figure 2.2



Since y is a constant $(y=b)$, we can think the intersection in x-z coordinate. (which is similar to ordinary derivative, so use same idea)

The slope of secant line P_1P_2 is:

$$m = \frac{f(x,b) - f(a,b)}{x - a}$$

to find the slope at $x=a$ (P_1), we let $x \to a$, so $x-a$ approaches 0, and $f(x,b) - f(a,b)$ approaches a number.

$$f_x(a,b) = \lim_{x \to a} \frac{f(x,b) - f(a,b)}{x - a}.$$

This is the slope at $x=a$, also the limit definition of partial derivative.

Note: this slope is negative b/c $f(a,b) > f(x,b)$ which is consistent with the graph.

Note: $f_x(a,b)$ is the slope at $x=a$, here $y=b$ is a constant.

lesson 9 Activities. (Part II)

4. Let $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a two-variable, real-valued function.

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

A. Find $f_x(x,y)$ and use this result to calculate $f_x(2,1)$

$\boxed{\text{Solution:}}$ Derivative with respect to x, hold y as constant.

$$f_x(x,y) = 3x^2 + 2xy^3$$

$$f_x(2,1) = 3(2)^2 + 2(2)(1)^3 = \boxed{16}$$

B. Find $f_y(x,y)$ and use this result to calculate $f_y(2,1)$

$\boxed{\text{Solution:}}$

$$f_y(x,y) = 3x^2y^2 - 4y$$

$$f_y(2,1) = 3(2)^2(1)^2 - 4(1) = \boxed{8}$$

C. Find the higher-order derivatives f_{xx}, f_{yx}, f_{xy}, and f_{yy}.

$\boxed{\text{Solution:}}$

$f_x(x,y) = 3x^2 + 2xy^3$ $\quad$ $f_y(x,y) = 3x^2y^2 - 4y$

$f_{xx} = 6x + 2y^3$ $\quad$ $f_{yy} = 6x^2y - 4$

$f_{xy} = \boxed{6xy^2}$ $\quad$ $f_{yx} = \boxed{6xy^2}$

※ Notice: $f_{xy} = f_{yx}$ (Clairaut's Theorem)

5. Let $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a two-variable, real-valued function given by: $f(x,y) = \sin\left(\frac{x}{1+y}\right)$

Find $f_x(x,y)$ and $f_y(x,y)$

Notice:
① apply chain rule
② hold y as a constant.

$\boxed{\text{Solution:}}$

$$f_x(x,y) = \boxed{\frac{1}{1+y} \cdot \cos\left(\frac{x}{1+y}\right)}$$

$$f_y(x,y) = [x \cdot (1+y)^{-1}]' \cdot \cos\left(\frac{x}{1+y}\right)$$

$$= \boxed{-\frac{x}{(1+y)^2} \cdot \cos\left(\frac{x}{1+y}\right)}$$

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#2 continue...

B. Explain how this definition of $f_x(a,b)$ relates to Path1 for taking multivariable limits.

Solution:

When we take path 1 for multivariable function, we control y as a constant, and think $z = f(x, y)$ as a single variable function WRT x, this is the same idea when we derive the definition of $f_x(a,b)$.

C. Using proper algebraic substitution, create limit definition of the partial derivatives in derivative notation given by:

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

Solution: From (A) we have:

$$f_x(a,b) = \lim_{x\to a} \frac{f(x,b) - f(a,b)}{x-a}$$

(same as: $x-a \to 0$)

let $h = x-a$,

$h = x-a$

$x = h+a$

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

$\begin{cases} x-a \to 0 \\ h \to 0 \end{cases}$

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6. Let $f: D \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a three-variable, real-valued function

$$f(x,y,z) = e^{xy} \cdot \ln(z)$$

Find $f_x(x,y,z)$, $f_y(x,y,z)$, $f_z(x,y,z)$.

$\boxed{\text{Solution:}}$

Note: this is a constant term.

$$f_x(x,y,z) = \boxed{(y \cdot \ln z)\, e^{xy}}$$

$$f_y(x,y,z) = \boxed{(x \cdot \ln z)\, e^{xy}}$$

$$f_z(x,y,z) = \boxed{(e^{xy}) \cdot \frac{1}{z}}$$

Recall: $(e^{2x})' = 2e^{2x}$

Recall: $[\ln(z)]' = \frac{1}{z}$.

7. Let $f: D \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a three-variable, real-valued function

$$f(x,y,z) = 3xy \arcsin(yz)$$

Find $f_x(x,y,z)$, $f_y(x,y,z)$ and $f_z(x,y,z)$

$\boxed{\text{Solution:}}$

$$f_x(x,y,z) = \boxed{\overbrace{\arcsin(yz)}^{\text{constant}} \cdot 3y}$$

$$f_y(x,y,z) = 3x \cdot \arcsin(yz) + 3xy \cdot \frac{1}{\sqrt{1-(yz)^2}} \cdot (yz)'$$

$$= \boxed{3x \cdot \arcsin(yz) + \frac{3xyz}{\sqrt{1-y^2z^2}}}$$

Recall: Product Rule

$(f \cdot g)' = f' \cdot g + f \cdot g'$

$(\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}$

8. Consider relation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

Assume $z = z(x,y)$, use implicit differentiation to find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$

W.R.T with respect to

Notice: z is a function of x.

$\boxed{\text{Solution}}$: take derivative of each term WRT x.

$$3x^2 + 3z^2 \cdot \frac{\partial z}{\partial x} + \underbrace{\left(6y \cdot z + 6xy \cdot \frac{\partial z}{\partial x}\right)}_{\text{product rule, } z = z(x)} = 0$$

(detailed steps on following page).

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detailed steps for #8.

$$3x^2+3z^2\cdot\frac{\partial z}{\partial x}+6yz+6xy\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial x}(3z^2+6xy)=-3x^2-6yz$$

$$\frac{\partial z}{\partial x}=\boxed{\frac{-3x^2-6yz}{3z^2+6xy}}$$

To find $\frac{\partial z}{\partial y}$, take derivatives of each term WRT y,

$$3y^2+3z^2\cdot\frac{\partial z}{\partial y}+\left(6x\cdot z+6xy\cdot\frac{\partial z}{\partial y}\right)=0.$$

$$\frac{\partial z}{\partial y}(3z^2+6xy)=-3y^2-6xz$$

$$\frac{\partial z}{\partial y}=\boxed{\frac{-3y^2-6xz}{3z^2+6xy}}$$

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