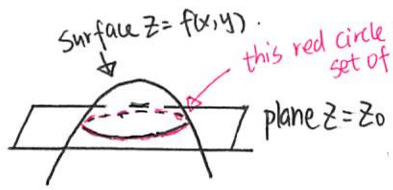
yanru chen

1. Suppose $z=f(x,y)$ is a two variable function, $z_0 \in \mathbb{R}$ in the range of f.

(a) What is the definition of Contour Curve $C_{z_0}(f)$?

Solution: The contour curve is the set of all ordered triplets (x, y, z_0) such that $f(x,y)=z_0$, notation:

$$C_{z_0}(f) = \{(x, y, z_0) : z_0 = f(x,y), \text{ for } (x,y) \in D\}.$$



this red circle is the set of all triplets satisfied $f(x,y)=z_0$. (contour curve).

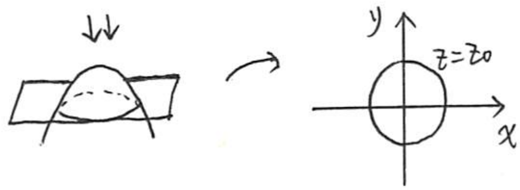
intersection of surface $z=f(x,y)$ & plane $z=z_0$

(b) What is the definition of the level curve $L_{z_0}(f)$?

Solution: The level curve is the set of input (x,y) in the domain of f that produce the contour curve $C_{z_0}(f)$, notation:

$$L_{z_0}(f) = \{(x,y) : z_0 = f(x,y), (x,y) \in D\}$$

Think: projection of $C_{z_0}(f)$ onto xy plane.



(c) Describe the similarities and differences between $L_{z_0}(f)$ and $C_{z_0}(f)$

Solution: similarity: they have same x, y value corresponding to a certain z_0.

differences: $C_{z_0}(f) \subseteq \mathbb{R}^3$ ordered triplets;

$L_{z_0}(f) \subseteq \mathbb{R}^2$ ordered pairs (in domain).

(d) Describe how level curves are related to contour curves?

$\boxed{\text{Solution}}$: they are two different ways to represent the set of value that satisfied $z_0 = f(x,y)$.

level curves is the projection of corresponding contour curve into xy plane.

The level curves is the set of input in the domain that produce the contour curve.

2. Find the domain and range of $f(x,y) = \sqrt{80-5x^2-5y^2}$

$\boxed{\text{Solution}}$: let $z = f(x,y)$

Recall: Domain is the set of input ((x,y) in this case.) that satisfy the function.

$\text{Dom}(f) = D \subseteq \mathbb{R}^2$ for $z = f(x,y)$

For $\sqrt{a}$, $a \geq 0$.

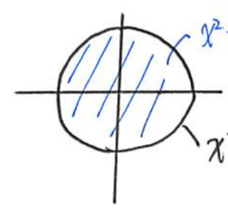
$\Rightarrow 80 - 5x^2 - 5y^2 \geq 0$

$\Rightarrow x^2 + y^2 \leq 16$

$\Rightarrow \text{Dom}(f) = \{(x,y) : x^2+y^2 \leq 16\}$

Note:
$\sqrt[n]{a}$
when $n=2$,
radicand
$a \geq 0$.

$\Rightarrow$ Dom(f) is the disk with radius $r = \sqrt{16} = 4$



Recall: Range is the set of all possible output for every input (x,y).

$\text{Rng}(f) = \{z : z = f(x,y) \text{ for } (x,y) \in \text{Dom}(f)\}$.

$z = f(x,y) = \sqrt{80-5x^2-5y^2}$

$\Rightarrow 80 - 5x^2 - 5y^2 \geq 0$ (since $\sqrt{a}$, $a \geq 0$)

$\Rightarrow 80 \geq 5x^2 + 5y^2 \geq 0$ (because $5x^2+5y^2 \geq 0$)

$\Rightarrow -80 \leq -5x^2 - 5y^2 \leq 0$ (if $a<x<b$, $\Rightarrow -a > -x > -b$)

$\Rightarrow 0 \leq 80 - 5x^2 - 5y^2 \leq 80$ (add 80 to all three parts).

$\Rightarrow 0 \leq \sqrt{80-5x^2-5y^2} \leq \sqrt{80} = 4\sqrt{5}$

$\Rightarrow \text{Rng}(f) = [0, 4\sqrt{5}]$.

1/23 2:50PM – 3:22 PM

3. Graph several level curves of $z = x^2 + \frac{y^2}{9}$.

$\boxed{\text{Solution:}}$

Recall: level curve is the set of input (x,y) in domain for certain z_0.

step1: pick some z_0

$$x^2 + \frac{y^2}{9} = \frac{x^2}{1^2} + \frac{y^2}{3^2} = z_0$$

Think this as a ellipse

$z_0 \subseteq \mathbb{R}$ (+) or (0)

▷ Let $z_0 = 1$,

$$\frac{x^2}{1^2} + \frac{y^2}{3^2} = 1$$

this is an ellipse with x-semiaxis 1 and y-semi axis 3. Center $(0,0)$

▷ let $z_0 = 3^2$,

$$\frac{x^2}{1^2} + \frac{y^2}{3^2} = 3^2$$

$$\Rightarrow \frac{x^2}{3^2} + \frac{y^2}{9^2} = 1$$

this is an ellipse with x-semiaxis 3 and y-semiaxis 9 Center $(0,0)$

▷ let $z_0 = 2^2$,

$$\frac{x^2}{1^2} + \frac{y^2}{3^2} = 2^2$$

$$\Rightarrow \frac{x^2}{2^2} + \frac{y^2}{6^2} = 1$$

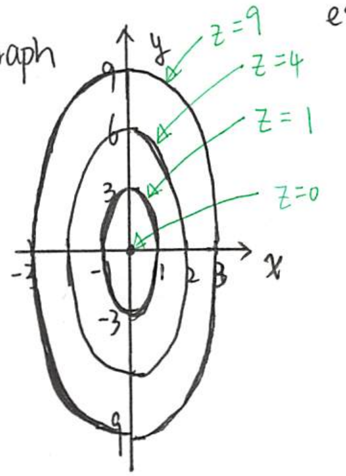
this is an ellipse with x-semiaxis 2 and y-semiaxis 6 Center $(0,0)$

▷ let $z_0 = 0$

$$\frac{x^2}{1^2} + \frac{y^2}{3^2} = 0$$

this is a point $(0,0)$ because $x^2 \geq 0$, $y^2 \geq 0$ only $(0,0)$ satisfy this equation.

step2: graph



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4. Identify trace of surface $\frac{x^2}{6}+24y^2+\frac{z^2}{24}-6=0$ in the plane $y=0$.

$\boxed{\text{Solution}}$:

Q: Is trace in $\mathbb{R}^2$ or in $\mathbb{R}^3$?

Let $y=0$,

$$\frac{x^2}{6}+\frac{z^2}{24}-6=0$$

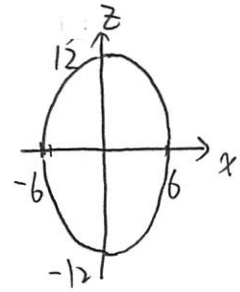
$$\Rightarrow \frac{x^2}{6}+\frac{z^2}{24}=6$$

$$\Rightarrow \frac{x^2}{36}+\frac{z^2}{144}=1$$

$$\Rightarrow \frac{x^2}{6^2}+\frac{z^2}{12^2}=1$$

the trace is an ellipse with x- semiaxis 6 and y- semi axis 12 center at $(0,0)$ in the <u>xz plane</u>.

(or say plane $y=0$)



5. Consider the function $z=x^2+y^2-16$

find a parametric equation to the tangent line of level curve $L_0(f)$ at point $(-2\sqrt{2},-2\sqrt{2})$. Graph this level curve and its tangent line

$\boxed{\text{Solution}}$: Note: when $x=-2\sqrt{2}$, $y=-2\sqrt{2}$, $z=(-2\sqrt{2})^2+(-2\sqrt{2})^2-16=0$

Idea: For an parametric equation of a line, we need a point (given) and a slope (y' or say $\frac{dy}{dx}$).

$$\frac{d}{dx}\left[x^2+y^2-16\right]=\frac{d}{dx}[0]$$

$$\Rightarrow \frac{d}{dx}[x^2]+\frac{d}{dx}[y^2]-\frac{d}{dx}[16]=0$$

apply chainrule

$$\Rightarrow 2x+\underline{2y\cdot y'}-0=0$$

solve for y': $y'=\frac{-x}{y}=\frac{-2\sqrt{2}}{2\sqrt{2}}=-1$

Slope at $(-2\sqrt{2},-2\sqrt{2})$ is -1

write slope in vector form: $\vec{v}=\langle -1,1\rangle$

Q: Dose $\langle 1,-1\rangle$ also true?

Recall: equation of line $\vec{r}(t)=\vec{r}_0+t\cdot\vec{v}$

$\vec{r}_0=\langle -2\sqrt{2},-2\sqrt{2}\rangle$

So equation of tangent line:

$$\vec{r}(t)=\langle -2\sqrt{2},-2\sqrt{2}\rangle+\langle -1,1\rangle\cdot t$$

$$\begin{cases} x(t)=-2\sqrt{2}-t \\ y(t)=-2\sqrt{2}+t \end{cases}$$

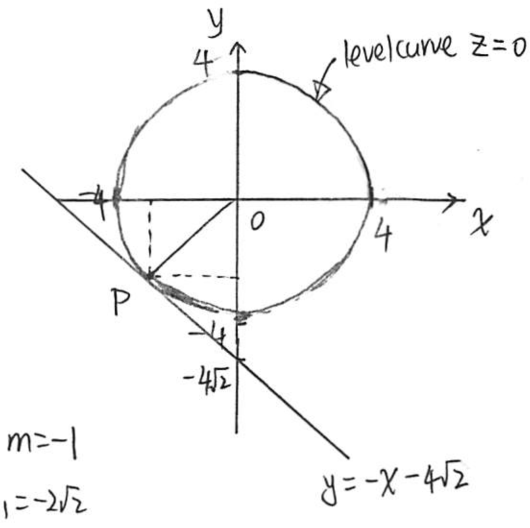
Question 4 continue

Graph:

- level curve $x^2+y^2-16=0$

$\Rightarrow x^2+y^2=16$

This is a circle with radius 4, center $(0,0)$



- tangent line

let $y=m(x-x_1)+y_1$ m is slope, $m=-1$

$x_1=-2\sqrt{2}$, $y_1=-2\sqrt{2}$

$\Rightarrow y=-1(x+2\sqrt{2})-2\sqrt{2}$

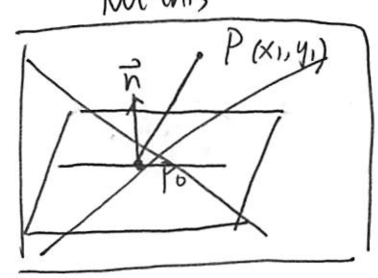
$=-x-4\sqrt{2}$

verify: when $x=-4\sqrt{2}$, $y=0$ as graph showed.

$P(-2\sqrt{2},-2\sqrt{2})$ $\overline{OP}^2=(-2\sqrt{2})^2+(-2\sqrt{2})^2=16$.

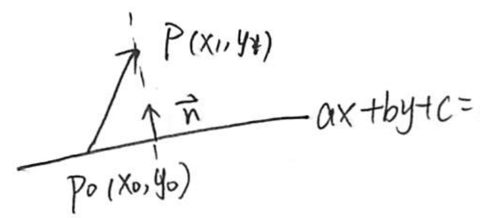
5. use scalar projection to show distance from $P(x_1,y_1)$ to line $ax+by+c=0$ is $\frac{|ax+by+c|}{\sqrt{a^2+b^2}}$

Not this



~~Not done~~.

distance from P to l: $ax+by+c=0$ is same as



$$\|\text{proj}_{\vec{n}}\overrightarrow{P_0P}\|_2=\left\|\frac{\overrightarrow{P_0P}\cdot\vec{n}}{\|\vec{n}\|_2}\right\|_2$$

for the line slope $=-\frac{a}{b}$

$$=\left\|\frac{\langle x_1-x_0,\ y_1-y_0\rangle\cdot\langle a,b\rangle}{\sqrt{a^2+b^2}}\right\|_2$$

$\vec{n}\perp\vec{v}$

we know if two line $\perp$, product of the two slope is -1

so slope of $\vec{n}$ is $\frac{b}{a}$.

$\vec{n}=\langle b,a\rangle$.

$$=\frac{|ax_1-ax_0+by_1-by_0|}{\sqrt{a^2+b^2}}$$

$$=\frac{|ax_1+by_1-(ax_0+by_0)|}{\sqrt{a^2+b^2}}$$

$$=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

Note:

$ax+by+c=0$

$ax_0+by_0+c=0$

$\Rightarrow c=-(ax_0+by_0)$

1/23 4:30pm - 4:40pm.