lesson 14 Lagrange Multipliers

Yanm chen.

1. Review the Parallel Gradient Theorem & Lagrange Multiplier Procedure:

A. What is a constrained optimization problem? what general form?

$\boxed{\text{solution}}$

Find max / min value of $f(x,y)$ subject to restriction that x, y lies on a constrained curve C in xy plane ($g(x,y)=0$)

General form:

$$\max_{(x,y)\in\mathbb{R}^2} f(x,y) \quad \text{subject to } g(x,y)=0$$

$$\min_{(x,y)\in\mathbb{R}^2} f(x,y) \quad \text{subject to } g(x,y)=0$$

B. How are constrained optimization problem different than multivariable optimization (lesson 13)

In lesson 13, Multivariable optimization problem is lookiy for "~~global" max / min, in other words~~, the max / min of the whole surface.

Constrained optimization problem is looking for the max / min of $f(x,y)$ on a constrained curve.

2/23 9:48 AM – 10:00 AM.

12 min

C. state, from memory, the Parallel Gradient Theorem.

$\boxed{\text{Solution}}$ Given function $f(x,y)$, the constrained max / min happened at $P(a,b)$ such that

$\nabla f(a,b) = \lambda \cdot \nabla g(a,b)$

The constrained curve C is given by $g(x,y)=0$, as level curve of $g(x,y)$.

D. what is the geometric interpretation of this theorem? How does this theorem relate to constrained curve $g(x,y)=0$ to the level curves of $f(x,y)$.

$\boxed{\text{Solution}}$ $\nabla f(a,b) \;//\; \nabla g(a,b)$

I skipped Q 2-4 here, please see (website)

A. Lesson 14 : Jeff's Handwritten Lesson Notes (.pdf).

#2 page 12

#3 page 15

#4 page 23

2/25 10:00 am - 10:10 am. 10min

5. Chapter 12 Review 93 (Brigg's)

Find the maximum & minimum values of

$$f(x,y) = 2x + y + 10$$

subject to constraint $2(x-1)^2 + 4(y-1)^2 = 1$

$\boxed{\text{Solution}}$

$f(x,y) = 2x+y+10$ this is a plane in 3D

$g(x,y) = 2(x-1)^2 + 4(y-1)^2 - 1$

Let $g(x,y) = 0$, $\dfrac{(x-1)^2}{\frac{1}{2}} + \dfrac{(y-1)^2}{\frac{1}{4}} = 1$

this is a ellipse in xy plane (in 2D).

Apply Lagrange Multiplier in two variables,

Eq1 : $f_x(x,y) = 2 = \lambda \cdot 4(x-1) = \lambda \cdot g_x(x,y)$

Eq2 : $f_y(x,y) = 1 = \lambda \cdot 8(y-1) = \lambda \cdot g_y(x,y)$

Eq3 : $g(x,y) = 2(x-1)^2 + 4(y-1)^2 - 1 = 0$

this is easier to calculate.

From $\dfrac{\text{Eq1}}{\text{Eq2}}$: $\dfrac{4(x-1)}{8(y-1)} = 2 \Rightarrow y = \dfrac{x+3}{4}$ or $x = 4y-3$

plug in Eq3: $2(x-1)^2 + 4(y-1)^2 - 1 = 0$

$$\Rightarrow 2(4y-3-1)^2 + 4(y-1)^2 = 1$$

$$2(4y-4)^2 + 4(y-1)^2 = 1$$

$$32(y-1)^2 + 4(y-1)^2 = 1$$

$$(y-1)^2 = \frac{1}{36}$$

$$y-1 = \pm \frac{1}{6}$$

$y_1 = \frac{7}{6}$, $y_2 = \frac{5}{6}$.

lesson 14 page 3/4

2/25 10:10AM – 11:00AM.

50min

when $y = \frac{7}{6}$, $x = 4(\frac{7}{6}) - 3 = \frac{5}{3}$

when $y = \frac{5}{6}$, $x = 4(\frac{5}{6}) - 3 = \frac{1}{3}$

$f(\frac{5}{3}, \frac{7}{6}) = 2 \times \frac{5}{3} + \frac{7}{6} + 10 = \frac{87}{6} = \frac{29}{2}$.

$f(\frac{1}{3}, \frac{5}{6}) = 2 \times \frac{1}{3} + \frac{5}{6} + 10 = \frac{23}{2}$

So the max value is $\frac{29}{2}$ minimum is $\frac{23}{2}$.

6. Find the max & min value of $f(x,y) = y^2 - 4x^2$ subject to constraint $x^2 + 2y^2 = 4$.

$\boxed{\text{Solution}}$ Let $g(x,y) = x^2 + 2y^2 - 4$

$\nabla f = \langle -8x, 2y \rangle$ $\nabla g = \langle 2x, 4y \rangle$

Using method of Lagrange multipliers in two variable.

$$\begin{cases} -8x = \lambda \cdot 2x & \text{Eq 1} \\ 2y = \lambda \cdot 4y & \text{Eq 2} \\ x^2 + 2y^2 - 4 = 0 & \text{Eq 3.} \end{cases}$$

Eq 1 / Eq $\Rightarrow$ $\frac{2x}{4y} = \frac{-8x}{2y}$ Simplify: $xy = -8xy$

this will only true when $xy = 0$.

Case 1: $x = 0$, by Eq 3, $2y^2 = 4$, $y = \pm\sqrt{2}$

we have $(0, \sqrt{2})$, $(0, -\sqrt{2})$

Case 2: $y = 0$, by Eq 3, $x^2 = 4$, $x = \pm 2$,

we have $(2, 0)$, $(-2, 0)$

Plug in $f(x,y)$. at $(2,0), (-2,0)$ $f = -16$ min.

at $(0, \sqrt{2})$ $(0, -\sqrt{2})$ $f = 2$ max.