Name : ____________________ **Class Number:** ____

Math 1C: MD Lesson 13 Suggested Problems

Theoretic Problems: Discussed in-class

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1. **Review of the method of completing the squares**

   A. Write each of the following perfect squares as the equivalent trinomial in the form $x^2 + 2bx + c$. Show your steps.

   i. $(x-4)^2$ ii. $(x+3)^2$ iii. $(x+11)^2$ iv. $\left(x - \frac{7}{2}\right)^2$

   Make sure to specifically find the values of the coefficients b and c.

   B. For each of the problems above, write the equivalent expressions in the form

   $$(x+d)^2 = x^2 + 2bx + c$$

   Please specifically identify the values of the coefficients d, b, and c.

   C. Look back on the work you finished in problem 2 above. What pattern do you notice? Specifically, how are the coefficients d, b, and c related to each other?

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2. **Review of quadratic functions** Consider the standard equation for a quadratic polynomial, given by

   $$f(x) = ax^2 + bx + c$$

   Use this general form to answer each of the following problems

   A. What do each of the coefficients $a, b, c \in \mathbb{R}$ do to the graph of this function?

   B. Use the method of completing the square to derive the quadratic formula.

   C. Use the method of completing the square to derive the vertex form of the quadratic function

   $$f(x) = a(x-h)^2 + k$$

   Your derivation should include an explicit formula for h and k in terms of a, b and c.

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3. **Derive the second ordinary derivative test** Suppose we are given a function $f : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ with continuous second derivative on the domain D. Let $a \in D$ be a given constant with point $(a, f(a))$ on the graph of our function. .Using this information, please:

   A. State, from memory, the second ordinary derivative test including all three conditions.

   B. Write the equation for the first-order Taylor series approximation $T_1(x)$ of this function at the point $(a, f(a))$. In other words, please write the equation to the tangent line to f at the point $(a, f(a))$. This is known as a local linear approximation of the function f at the point $(a, f(a))$.

C. Write the equation for the second-order Taylor series approximation $T_2(x)$ of this function at the point $(a, f(a))$. In other words, write the equation for the tangent parabola at point $(a, f(a))$. This is known as a local quadratic approximation of the function f at the point $(a, f(a))$.

D. Explain how the second ordinary derivative test is related to the idea of the tangent line $T_1(x)$ and the tangent parabola $T_2(x)$. In particular, reinterpret the three conditions of the second ordinary derivative test using language about the graphs of these tangent polynomials. For example, what is the slope of the tangent line at a local minimum point? Does the tangent parabola point upwards or downwards at such a minimum? By answering these type of questions for each of the conditions, you will be building a strong geometric interpretation associated with the second ordinary derivative test which you can then use to guide your understanding of the second partial derivative test.

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4. **Derive the second partial derivative test** Suppose we are given a function $f : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ with continuous second derivative on the domain D. Let $(a, b) \in D$ be a given constant with point $(a, b, f(a, b))$ on the graph of our function. Using this information, please:

A. State, from memory, the second partial derivative test including all four conditions.

B. Write the equation for the first-order Taylor series approximation $T_1(x, y)$ of f at point $(a, b, f(a, b))$. In other words, please write the equation to the tangent plane to f at point $(a, b, f(a, b))$. This is known as a local linear approximation of the function f at the point $(a, b, f(a, b))$.

C. Write the equation for the second-order Taylor series approximation $T_2(x, y)$ of f at the point $(a, b, f(a, b))$. In other words, write the equation for the tangent quadratic surface at point $(a, b, f(a, b))$. This is known as a local quadratic approximation of the function f at the point $(a, b, f(a, b))$.

D. Combine the fact that we want to look at points on the surface where $\nabla f(x, y) = \mathbf{0}$ with algebra to translate your equation for $T_2(x, y)$ into the form

$$T_2(x, y) = \frac{1}{2}\left(ax^2 + 2bxy + cy^2\right) + k_0.$$

In your work, specifically identify the values of a, b and c in terms of the second-order partial derivatives f_{xx}, f_{yy}, and f_{yy}.

E. Use the method of completing the square to show the following:

$$ax^2 + 2bxy + cy^2 = a\left(x - \frac{b}{a}y\right)^2 + \frac{ac - b^2}{a}y^2.$$

Using this factorization, explain how to determine the behavior of the local quadratic approximation $T_2(x, y)$ based on the values of the coefficients a and $ac - b^2$.

F. Explain how the second ordinary derivative test is related to the idea of the tangent line $T_1(x)$ and the tangent parabola $T_2(x)$. In particular, reinterpret the three conditions of the second ordinary derivative test using language about the graphs of these tangent polynomials. For example, what is the slope of the tangent line at a local minimum point? Does the tangent parabola point upwards or downwards at such a minimum? By answering these type of questions for each of the conditions, you will be building a strong geometric interpretation associated with the second ordinary derivative test which you can then use to guide your understanding of the second partial derivative test.

Problems Solved in Jeff's Handwritten Notes

5. Quick check 12.8.1 p. 939

6. Example 12.8.1 p. 940

7. Example 12.8.3 p. 942

8. Example 12.8.9 p. 948
   A. Solve this problem using the second partial derivative test.

   B. Solve this problem using orthogonal projections.
   C. Check that your two solutions are the same.

Suggested Problems

9. Find the point(s) on the plane $x + y + z = 4$ nearest the point $P(0, 3, 6)$.

10. Find the point on the curve $y = x^2$ nearest the line $y = x - 1$. Identify the point on the line.

Optional Challenge Problems

8. Exercise 12.8.70 p. 950
9. Exercise 12.8.71 p. 950
10. Exercise 12.8.76 p. 951