lesson 13 Multivariable optimization

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214 + P8

1. Review the method of completing the squares.

A. Write perfect squares as equivalent trinomial in form $x^2+2bx+c$

(i) $(x-4)^2$

$\boxed{\text{solution}}$ $x^2-8x+16$ $b=-4$, $c=16$

(ii) $(x+3)^2$

$\boxed{\text{Solution}}$ x^2+6x+9 $b=3$, $c=9$

(iii) $(x+11)^2$

$\boxed{\text{Solution}}$ $x^2+22x+121$ $b=11$, $c=121$

(iv) $(x-\frac{7}{2})^2$

$\boxed{\text{solution}}$ $x^2-7x+\frac{49}{4}$ $b=-\frac{7}{2}$, $c=\frac{49}{4}$

B. For each problem above, write equivalent expression in form

$$(x+d)^2 = x^2+2bx+c$$

$\boxed{\text{Solution}}$

(i) $(x-4)^2 = x^2-8x+16$ $d=-4$, $b=-4$, $c=(-4)^2$

(ii) $(x+3)^2 = x^2+6x+9$ $d=3$, $b=3$, $c=(3)^2$

(iii) $(x+11)^2 = x^2+22x+121$ $d=11$, $b=11$, $c=(11)^2$

(iv) $(x-\frac{7}{2})^2 = x^2-7x+\frac{49}{4}$ $d=-\frac{7}{2}$, $b=-\frac{7}{2}$, $c=(-\frac{7}{2})^2$

C. what pattern do you notice?

$\boxed{\text{Solution}}$ $b=d$, $c=d^2=b^2$

2/18 8:50PM – 8:57PM 7min

2. Review of quadratic functions

Consider standard equation for a quadratic polynomial

$$f(x) = ax^2+bx+c$$

A. what do each coefficient a, b, c do to graph of this function?

Solution • a determine the "shape" of parabola, $0<|a|<1$, parabola is wider; $|a|>1$ parabola is thinner; when $a>0$, parabola open upwards, when $a<0$, parabola open downwards. $a=0$ makes graph a line.

• b affect the location of vertex.
when $b=0$, vertex lies on y-axis; when a, b have same sign ($\pm$) vertex lies on right of y-axis; when a, b have different sign vertex on left.
(Note: b does Not change shape of parabola)

• c affect the vertical shift of parabola.
$c>0$ shift up; $c<0$ shift down.

Ref: Exploring Parabolas By Joshua Singer

B. Use the method of completing the square to derive the quadratic formula.

Solution Recall: quadratic formula is: $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Set $ax^2+bx+c=0$, we want to find x that makes it true.

we hope to transform this into form:

$$k(x+m)^2 = n, \quad k, m, n \in \mathbb{R}$$

so that we could find x by $x = -m \pm\sqrt{\frac{n}{k}}$

In other words, we want to have only one term with x.

2/18 8:57 – 9:26 PM
30 min

Continue #2

$$ax^2+bx+c=0$$

$$\Rightarrow a\left(x^2+\frac{b}{a}x\right)+c=0$$

Recall from #1 $\quad x^2+2bx+c=(x+d)^2, \quad d=b, \; c=b^2$

Note: Remember times a!

$$\Rightarrow a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right)-a\cdot\left(\frac{b}{2a}\right)^2+c=0$$

$$\Rightarrow a\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a}-c$$

$$\Rightarrow a\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a}$$ Eq (I). use in Part C.

$$\Rightarrow \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$$\Rightarrow x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$\Rightarrow x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ Q.E.D.

c. Use the method of completing the square to derive the vertex form of the quadratic function. $f(x)=a(x-h)^2+k$

Derivation should include explicit formula for h & k in terms of a, b, c.

$\boxed{\text{Solution}}$ From #2 Part B we have:

$$f(x)=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}$$ (Eq I)

so we have $h=-\frac{b}{2a}, \; k=\frac{4ac-b^2}{4a}$ (Note the negative sign before k)

2/18 9:27 PM – 9:44 PM. 17 min

3. Derive second ordinary derivative test

Suppose we are given function $f: D \subseteq \mathbb{R} \to \mathbb{R}$ with continuous second derivative on domain D. let $a \in D$ be a given constant with point $(a, f(a))$ on the graph of our function.

A. State the second ordinary derivative test including all three conditions.

$\boxed{\text{Solution:}}$ Given $f'(a) = 0$ (first ordinary derivative)

if $f''(a) > 0$, $f(a)$ is local min ;

if $f''(a) < 0$, $f(a)$ is local max ;

if $f''(a) = 0$, inconclusive.

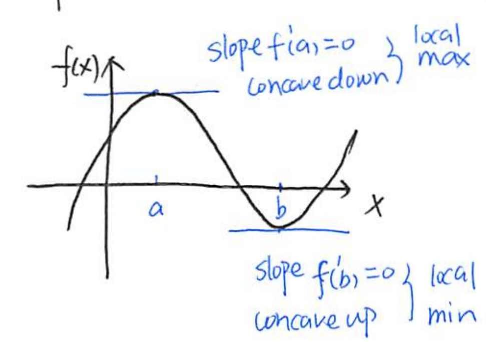
Note:

first ordinary derivative tells the slope at point $x=a$;

second ordinary derivative tells concave up / concave down.

slope $=0$, concave up $\Rightarrow$ local min ;

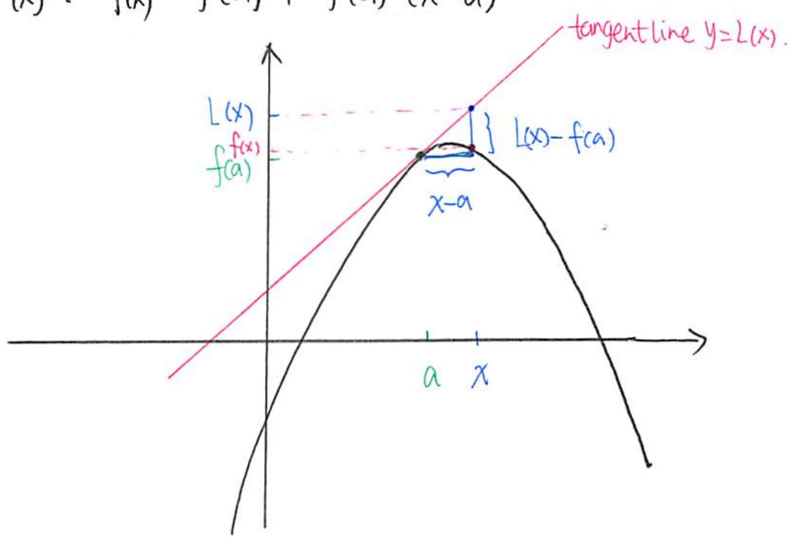
slope $=0$, concave down $\Rightarrow$ local max.



B. Write the equation for the first order Taylor series approximation $T_1(x)$ at $(a, f(a))$ AKA local linear approximation of f at $(a, f(a))$

$\boxed{\text{Solution}}$

$T_1(x)$: $f(x) = f(a) + f'(a)(x-a)$



Notice: we are using $L(x)$ to approximate $f(x)$

$\because L'(a) = \dfrac{f(x) - f(a)}{x-a} \Rightarrow L(x) = f(a) + f'(a)(x-a)$

☆ This is also slope of tangent line L

Then we set $f(x) = L(x)$ [APPROX!]

26min

2/18 9:45PM - 10:11 PM.

continue #3

c. Write the equation for second-order Taylor series approx $T_2(x)$ of f at $(a, f(a))$. AkA equation ~~of~~ for the tangent parabola at $(a, f(a))$; or local quadratic approx of f at point $(a, f(a))$

$\boxed{\text{solution}}$

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Notice: tangent parabola share $f(a)$, $f'(a)$ with linear approx.

Let $f(x) = a_0 + a_1(x-a) + a_2(X-2)^2$ ※: This is a parabola

$f''(x) = 2a_2$, $f''(a) = 2a_2$

$\Rightarrow a_2 = \frac{f''(a)}{2}$

Replace $f(a) = a_0$, $f'(a) = a_1$, $\frac{1}{2}f''(a) = a_2$,

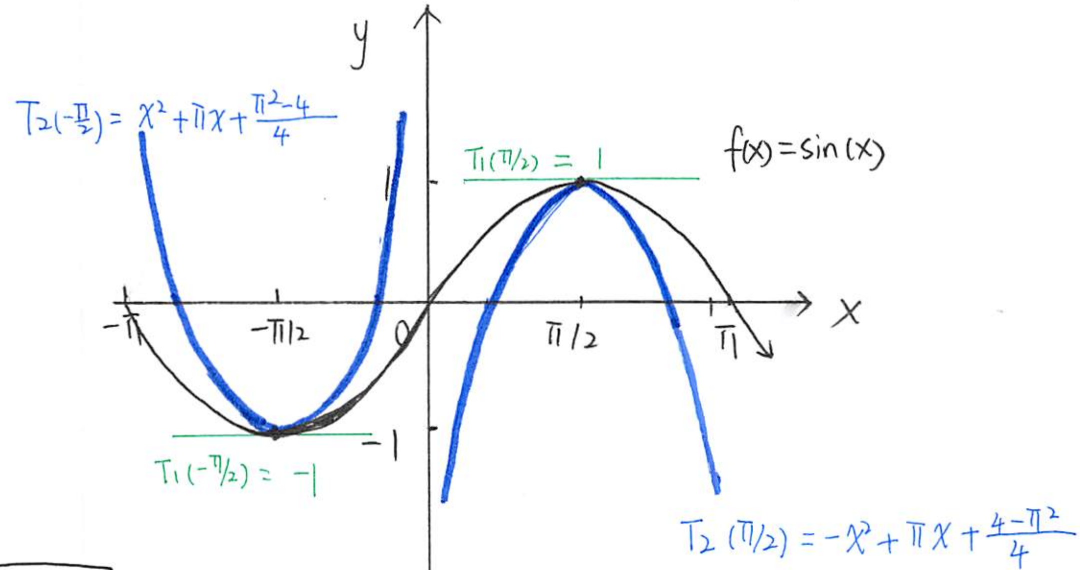
$$\Rightarrow T_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

2/19 7:38AM-7:55AM. 17min

continue #3

D. Explain how second ordinary derivative test is related to tangent line $T_1(x)$ & Tangent parabola $T_2(x)$.

Reinterpret the three conditions of second ordinary derivative test using language about the graph of tangent polynomials.



$\boxed{\text{Solution}}$:

Recall second ordinary derivative test

Given $f'(a) = 0$

$$\Rightarrow \begin{cases} f''(a) > 0 & \text{local min} \\ f''(a) < 0 & \text{local max} \\ f''(a) = 0 & \text{inconclusive.} \end{cases}$$

Attach :

Mathematica Plot P 7 (next)

derivation of $T_1(\pi/2)$ ... on same page.

From the graph of $f(x) = \sin(x)$

At $x = \pi/2$, local max, $T_1(\pi/2)$ has slope 0 ($T_1'(\pi/2) = 0$) $T_2(\pi/2)$ concave down ($T_2''(\pi/2) < 0$)

At $x = -\pi/2$, local min $T_2(\pi/2)$ has slope 0 ($T_1'(\pi/2) = 0$) $T_2(-\pi/2)$ concave up ($T_2''(-\pi/2) > 0$)

$\Rightarrow$ In conclusion,

tangent line ($T_1(x)$) has same $\boxed{\text{slope}}$ as $f(x)$;

tangent parabola ($T_2(x)$) has same $\boxed{\text{concavity}}$ as $f(x)$.

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2/19 11:13 AM – 11:40 AM

2/20 9:45 AM – 10:47 AM.

1h 30min

continue #3

In[18]:=
```
Plot[{Sin[x], 1, -1, -x^2 + Pi * x + (4 - Pi^2) / 4, x^2 + Pi * x + (Pi^2 - 4) / 4},
  {x, -Pi, Pi}, PlotRange → {-1.5, 1.5}]
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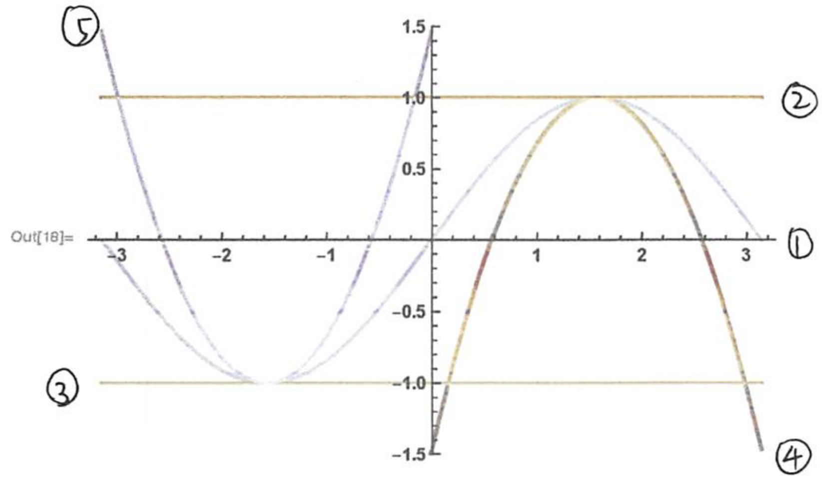
Out[18]=



Derive functions:

$f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

Note:
$\sin(\pi/2) = 1$
$\sin(-\pi/2) = -1$
$\cos(\pi/2) = 0$
$\cos(-\pi/2) = 0$

$T_1(a) = f(a) + f'(a)(x-a) = \sin(a) + \cos(a)(x-a)$

$T_2(a) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2$

$= \sin(a) + \cos(a)(x-a) - \sin(a)(x-a)^2$

so the equations are:

① $f(x) = \sin(x)$ original function

② $T_1(\pi/2) = \sin(\pi/2) + \underbrace{\cos(\pi/2)}_{\text{this is 0}}(x-\pi/2) = 1$

③ $T_1(-\pi/2) = \sin(-\pi/2) + \cos(-\pi/2)(x-\pi/2) = -1$

④ $T_2(\pi/2) = \sin(\pi/2) + \overset{=0}{\cancel{\cos(\pi/2)(x-\pi/2)}} - \sin(\pi/2)(x-\pi/2)^2$

$= -x^2 + \pi x + \frac{4-\pi^2}{4}$

⑤ $T_2(-\pi/2) = \sin(-\pi/2) + \overset{=0}{\cancel{\cos(-\pi/2)(x+\pi/2)}} - \sin(-\pi/2)(x+\pi/2)^2$

$= x^2 + \pi x + \frac{\pi^2-4}{4}$

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2/20 10:48–11:00AM
17min 11:30–11:35AM

4. Derive Second Partial Derivative test

Given $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ with continuous second derivative on Domain D.

Let $(a,b) \in D$, $(a, b, f(a,b))$ on the graph of the function.

A. state the second partial derivative test (4 conditions).

$\boxed{\text{solution}}$

Given $\nabla f(a, b) = 0$

Let $D = f_{xx}(a,b) f_{yy}(a,b) - f_{xy}^2(a,b)$

if $D > 0$, $\begin{cases} f_{xx}(a,b) > 0, & \text{there is min (local)} \\ f_{xx}(a,b) < 0, & \text{there is max.} \end{cases}$

if $D < 0$, there is saddle points.

if $D = 0$, inconclusive.

B. Write equation of first order Taylor series approx $T_1(x,y)$ at $(a, b, f(a,b))$. In other words, tangent plane of f at $(a,b,f(a,b))$. AKA local linear approx of f at $(a,b,f(a,b))$.

$\boxed{\text{Solution}}$

Recall tangent plane equation:

$$z - z_0 = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\Rightarrow z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

1/21 7:37 AM - 7:47 AM 10min

Continue #4

C. Write the equation for second order Taylor series approx $T_2(x,y)$ of f at point $(a, b, f(a,b))$. In other words, write the equation for the tangent quadratic surface at $(a, b, f(a,b))$. AKA local quadratic approximation of f at $(a, b, f(a,b))$.

$\boxed{\text{Solution}}$

$$T_2(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + \frac{f_{yy}(a,b)}{2}(y-b)^2 + f_{xy}(a,b)(x-a)(y-b)$$

D. Combine the fact that we want to look at points on surface where $\nabla f(x,y) = 0$ with algebra to translate your equation $T_2(x,y)$ into form

$$T_2(x,y) = \frac{1}{2}(ax^2 + 2bxy + cy^2) + k_0$$

Identify values of a, b, c in terms of f_{xx}, f_{yy}, f_{xy}.

$\boxed{\text{Solution}}$

replace a, b by x_0, y_0, to avoid confusion in this part...

$= 0$ since $\nabla f = 0$
$f_x(x_0,y_0) = f_y(x_0,y_0) = 0$.

Let $T_2(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \frac{1}{2} f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0)$

$\Rightarrow T_2(x,y) = f(x_0,y_0) + \frac{1}{2} f_{xx}(x_0,y_0)(x-x_0)^2 + \frac{1}{2} f_{yy}(x_0,y_0)(y-y_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0)$

Let $a = f_{xx}(x_0,y_0)$, $b = f_{xy}(x_0,y_0)$, $c = f_{yy}(x_0,y_0)$, $k_0 = f(x_0,y_0)$

$\Rightarrow T_2(x,y) = k_0 + \frac{1}{2} a(x-x_0)^2 + \frac{1}{2} c(y-y_0)^2 + b(x-x_0)(y-y_0)$

Let $X = x - x_0$, $Y = y - y_0$, (change base)

$\Rightarrow T_2(X, Y) = k_0 + \frac{1}{2}(aX^2 + 2bXY + cY^2)$

2/21 11:04 AM – 11:45 AM

continue #4

I think this should be "+"

E. Use the method of completing the square to show the following:

$$ax^2+2bxy+cy^2 = a(x - \frac{b}{a}y)^2 + \frac{ac-b^2}{a}y^2$$

Use this factorization, explain how to determine the behavior of the local quadratic approximation $T_2(x,y)$ based on the values of the coefficient a and $ac-b^2$.

Solution

(I) completing square:

$$\begin{aligned} ax^2+2bxy+cy^2 &= a(x^2+\frac{2b}{a}xy)+cy^2 \\ &= a(x^2+\frac{2b}{a}xy+(\frac{b}{a}y)^2)+cy^2-(\frac{b}{a})^2\cdot a\cdot y^2 \\ &= a(x+\frac{b}{a}y)^2+\frac{ac-b^2}{a}\cdot y^2 \end{aligned}$$

where $a=f_{xx}(x_0,y_0)$, $b=f_{xy}(x_0,y_0)$, $c=f_{yy}(x_0,y_0)$

(II) Since the square is always nonnegative,

- if $a>0$, $ac-b^2>0$, then $ax^2+2bxy+cy^2>0$

  $\Rightarrow f_{xx}(x_0,y_0)>0$, $f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-f_{xy}(x_0,y_0)>0$

  This is when we have local min

- if $a<0$, $ac-b^2>0$, then $ax^2+2bxy+cy^2<0$

  Note: $\frac{ac-b^2}{a}<0$, $a<0$, $ac-b^2>0$

  $\Rightarrow f_{xx}(x_0,y_0)<0$, $f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-f_{xy}(x_0,y_0)>0$

  This is when we have local max

- The other case is saddle point, where a and $\frac{ac-b^2}{a}$ has opposite sign.

  if $a>0$, $\frac{ac-b^2}{a}<0$, $ac-b^2<0$;

  if $a<0$, $\frac{ac-b^2}{a}>0$, $ac-b^2<0$.

  Either case $f_{xx}f_{yy}-f_{xy}<0$.

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2/25 8:45PM – 9:10PM

I think "partial" should be here.

F. Explain how the second ~~ordinary~~ partial derivative test is related to the idea of the tangent line $T_1(x)$ and tangent parabola $T_2(x)$. Reintepret the three conditions of the second partial derivative test using language about the graphs of these tangent polynomials.

$\boxed{\text{Solution}}$

when $\nabla f(a,b) = 0$, $f_x(a,b) = f_y(a,b) = 0$

$T_1(x,y) = f(a,b)$ which is a constant.

The tangent plane T_1 is parallel to the xy plane.

(a,b) might be max / min / saddle points.

$$D = f_{xx} f_{yy} - f_{xy}^2$$

If $D > 0$, and $f_{xx} > 0$

From Part (E), we know this is a local min

$T_2(x,y) \geq k_0$, where $k_0 \in \mathbb{R}$.

The tangent quadratic surface ~~is~~ point upwards;

If $D > 0$, and $f_{xx} < 0$

From Part (E), we know this is a local max

$T_2(x,y) \leq k_0$

The tangent quadratic surface point downwards.

9. Find the points on plane $x+y+z=4$ nearest the point $P(0,3,6)$

$\boxed{\text{Solution}}$

Notice one way to do this is projection.

Here let's use another way: partial derivative

Step: ① Construct a distance function

② use partial derivative test to find input that produce function min.

① distance function.

$x+y+z=4 \Rightarrow z = 4-x-y$

Distance of any given point on the plane to $P(0,3,6)$ is:

$$D = \sqrt{(x-0)^2+(y-3)^2+(4-x-y-6)^2}$$

Minimize D is same as minimize D^2 since $D \geq 0$

$$D^2 = x^2+(y-3)^2+(-x-y-2)^2$$

$$= x^2+(y-3)^2+(x+y+2)^2 \qquad \text{Note: } (-1)^2=1$$

$$= 2x^2+2y^2-2y+4x+13$$

② use partial derivative test to find min & the input value.

Let $f(x,y) = 2x^2+2y^2-2y+4x+13$

$\nabla f(x,y) = \langle 4x+4, 4y-2\rangle = 0 \quad x=-1,\ y=\frac{1}{2}$ → Note: $(-1, \frac{1}{2})$ is a critical point.

$f_{xx}f_{yy}-f_{xy}^2 = 4\times 4-0 = 16 > 0$

$\begin{cases} f_{xx} > 0 \\ f_{xx}f_{yy}-f_{xy}^2 > 0 \end{cases} \Rightarrow$ we have a local min at $(-1, \frac{1}{2})$.

$z = 4+1-\frac{1}{2} = \frac{9}{2}$

So the closet point is $(-1, \frac{1}{2}, \frac{9}{2})$.

2/25 9:34 – 9:55 PM.

10. Find the point on the curve $y = x^2$ nearest line $y = x-1$

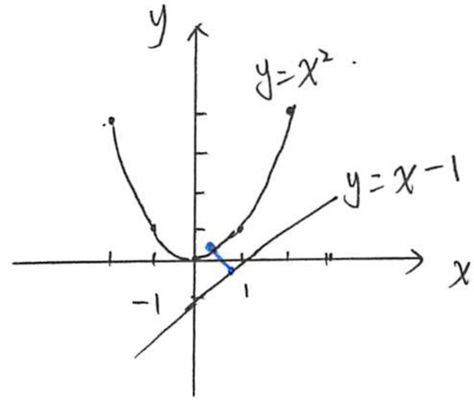
Identify the point on the line.

$\boxed{\text{Solution}}$

STEP:

① find distance function

② use partial derivative to find the minimum.



① To avoid confusion, change variable to the functions:

$y = x^2, \quad \Rightarrow f(u) = u^2 \qquad (u, u^2)$

$y = x-1 \quad \Rightarrow f(w) = w-1 \qquad (w, w-1)$

$$d = \sqrt{(w-1-u^2)^2 + (w-u)^2}$$

$$= \sqrt{w^2-2w-1 + u^4 - 2(w-1)u^2 + w^2 - 2wu + u^2}$$

$$= \sqrt{u^4 + 3u^2 + 2w^2 - 2wu^2 - 2wu - 2w + 1}$$

Let $D = d^2$,

$$D(u,w) = u^4 + 3u^2 + 2w^2 - 2wu^2 - 2u - 2w + 1$$

② Now use partial derivative to find the minimum point.

$$\nabla D(u,w) = \langle 4u^3 + 6u - 4wu - 2w,\ 4w - 2u^2 - 2u - 2 \rangle$$

In order to find the critical point, Set $\nabla D = \vec{0}$.

$$\Rightarrow \begin{cases} 4u^3 + 6u - 4wu - 2w = 0 & \text{Eq 1} \\ 4w - 2u^2 - 2u - 2 = 0 & \text{Eq 2} \end{cases}$$

continue #10

Solve for this group of equation:

Eq2: $4w = 2u^2 + 2u + 2 \Rightarrow w = \frac{1}{2}(u^2 + u + 1)$ Eq3

plug ~~in~~ Eq3 into Eq1 to replace w with u:

$4u^3 + 6u - 2u(u^2 + u + 1) - (u^2 + u + 1) = 0$

$\Rightarrow 4u^3 + 6u - 2u^3 - 2u^2 - 2u - u^2 - u - 1 = 0$

$\Rightarrow 2u^3 - 3u^2 + 3u - 1 = 0.$

One way to solve the cubic function with integer coefficient is:

$$r = \frac{s}{t}$$

where r is possible root, s is the factor of a_0 (-1 in this Eq),
t is factor of a_n (2 in this Eq).

So we have $s = \pm 1$, $t = \pm 2$

possible roots: $\pm\frac{1}{2}$, ± 1

Test root:

$r = 1 \quad 2 - 3 - 3 - 1 = 0$ ✗

$r = -1 \quad -2 - 3 - 1 - 1 = 0$ ✗

$r = \frac{1}{2} \quad 2(\frac{1}{8}) - 3(\frac{1}{4}) + 3(\frac{1}{2}) - 1 = \frac{1}{4} - \frac{3}{4} + \frac{3}{2} - 1 = 0$ ✓

$r = -\frac{1}{2} \quad 2(-\frac{1}{8}) - 3(\frac{1}{4}) + 3(-\frac{1}{2}) - 1 = -\frac{1}{4} - \frac{3}{4} - \frac{3}{2} - 1 = 0$ ✗

only $r = \frac{1}{2}$ is the root of the function $2u^3 - 3u + 3u - 1 = 0$.

So $u = \frac{1}{2}$,

Thus the point on $y = x^2$ that is nearest to $y = x - 1$ is

$(\frac{1}{2}, \frac{1}{4})$

☆ Last STEP is using <u>second partial derivative test</u>
to see if $(\frac{1}{2}, \frac{1}{4})$ is the min of distance function.
I will skip this step since I only have one critical point.

2/25 11:25AM–11:50AM.