lesson 12 Activities.

Tangent Plan & linerization.

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1. Derive the equation for a tangent plane

Given surface in $\mathbb{R}^3$ defined implicitly as a level surface using

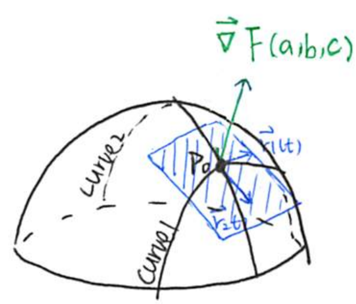
$$F(x, y, z) = 0$$

A. Derive the equation for the tangent plane to the level surface at point $\underline{(a, b, c)}$ on this surface. Make sure to mention the $\underline{\text{geometric interpretation}}$ of the gradient vector WRT tangent plane.

Solution

To define a plane, we need:

- a point on the plane
- normal vector to the plane



The point on tangent plane is: P_0 (a, b, c)

The normal vector to the tangent plan is: $\vec{n} = \vec{\nabla} F(a, b, C)$

(the prove of $\vec{n}$ is gradient in second page).

We Define a point $P(x, y, z)$ on the plane,

$\Rightarrow \vec{P_0P} = \langle (x-a), (y-b), (z-c) \rangle$

Since $\vec{n} \cdot \vec{P_0P} = 0$ (b/c normal vector always $\perp$ to any line in plane)

$\Rightarrow \langle F_x(a,b,c), F_y(a,b,c), F_z(a,b,c) \rangle \cdot \langle (x-a), (y-b), (z-c) \rangle = 0$

$\Rightarrow \boxed{F_x(a,b,c)\,(x-a) + F_y(a,b,c)\,(y-b) + F_z(a,b,c)\,(z-c) = 0}$

2/12 11:08AM – 11:20AM 12min

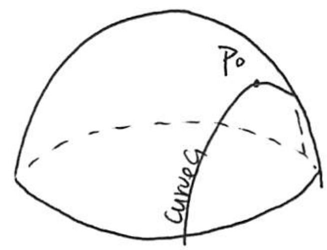
Prove gradient vector $\vec{\nabla} F(a,b,c)$ is the normal vector $\vec{n}$ at (a,b,c):

Assume we have <u>any</u> curve on the surface:

C_1: $\vec{r}(t) = \langle x(t),\ y(t),\ z(t) \rangle$

since C is on the surface,

we have:



$F(x(t),\ y(t),\ z(t)) = 0$

$\Rightarrow F(\vec{r}(t)) = 0$

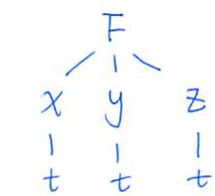
$\frac{d}{dt}[F(x(t),\ y(t),\ z(t))]$

$= \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \cdot \frac{dz}{dt}$

$= \langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$ dot product expand

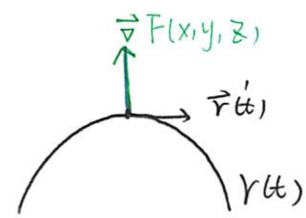
$= \langle F_x(x,y,z),\ F_y(x,y,z),\ F_z(x,y,z) \rangle \cdot \langle x'(t),\ y'(t),\ z'(t) \rangle$

$= \vec{\nabla} F(x,y,z) \cdot \vec{r}'(t)$



Recall: $F(x(t),\ y(t),\ z(t)) = 0$

$\frac{d}{dt}[F(x(t),\ y(t),\ z(t))] = \frac{d}{dt}[0] = 0$



$\Rightarrow \vec{\nabla} F(x,y,z) \cdot \vec{r}'(t) = 0$

In other words, $\vec{\nabla} F(x,y,z)$ always orthogonal to $\underbrace{\text{"direction" of curve } \vec{r}(t)}_{\vec{r}'(t)}$

B. Suppose we have $z = f(x,y)$ in $\mathbb{R}^3$. Derive equation for a tangent plane to this surface at $(a, b, f(a,b))$ & relate part A.

Solution:

From Part A we have:

$$F_x (x-a) + F_y (y-b) + F_z (z-c) = 0$$

Rewrite $z = f(x,y)$ as: $f(x,y) - z = 0$

$$\Rightarrow f_x (x-a) + f_y (y-b) + (-1)(z-c) = 0$$

since $c = f(a,b)$

$$\Rightarrow f_x (x-a) + f_y (y-b) - z + f(a,b) = 0$$

$$\Rightarrow z = f_x (x-a) + f_y (y-b) + f(a,b).$$

Note:

$$\frac{\partial}{\partial z}(-z) = -1$$

here z is independent of x & y

we rewrite $F(x,y,z) = 0$

as $f(x,y) - z = 0$

2/18 9:53AM – 10:43AM. 50min

2. Consider implicit relation for ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0$$

A. Find equation for tangent plane at $(0, 4, \frac{3}{5})$

$\boxed{\text{Solution}}$:

To find tangent plane, we use formula in #1 part A.

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0.$$

$$F_x(0,4,\tfrac{3}{5}) = \frac{2}{9}x = \frac{2}{9} \times 0 = 0$$
$$F_y(0,4,\tfrac{3}{5}) = \frac{2}{25}y = \frac{2}{25} \times 4 = \frac{8}{25}$$
$$F_z(0,4,\tfrac{3}{5}) = 2z = 2 \times \frac{3}{5} = \frac{6}{5}$$

$$\Rightarrow 0(x-0) + \frac{8}{25}(y-4) + \frac{6}{5}(z-\frac{3}{5}) = 0$$

$$\Rightarrow \boxed{\frac{8}{25}y + \frac{6}{5}z - 2 = 0}$$

B. Find any points on ellipsoid with a horizontal tangent plane.

To get a horizontal tangent plane, $\vec{n}$ is $\langle 0, 0, k \rangle$ (or $k \cdot \vec{j}$, $k \in \mathbb{R}$)
$\vec{j} = \langle 0, 0, 1 \rangle$.

when $x=0$, $y=0$, we have:

$$0 + 0 + z^2 - 1 = 0$$

$$\Rightarrow z = \pm 1$$

So there are two points on the ellipsoid $(0,0,1)$ & $(0,0,-1)$ that has a horizontal tangent plane.

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2/13 10:45AM – 11:00AM.
15 min

3. Find equation of tangent plane to elliptic paraboloid

$$z = f(x,y) = 2x^2 + y^2$$

at $(1,1,3)$

$\boxed{\text{Solution}}$:

We know the formula for tangent plane from #1 Part B.

$$z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$$

$$f_x(1,1) = 4x = 4$$

$$f_y(1,1) = 2y = 2$$

$f(a,b) = 3$ (Recall: $(a, b, f(a,b))$)

$$\Rightarrow z = 4(x-1) + 2(y-1) + 3$$

$$\Rightarrow \boxed{z = 4x + 2y - 3}$$

4. Define a two-variable, real-valued function

$$f(x,y) = \frac{5}{x^2+y^2}$$

Find linear approximation of f at $(-1,2,1)$

$\boxed{\text{Solution}}$

Recall Linear Approximation:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Notice this is also formula for <u>tangent plane</u> at $(a,b,f(a,b))$

11:06 – 11:26 AM 20min

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$f_x(-1,2) = \frac{\partial}{\partial x}\left[\frac{5}{x^2+y^2}\right] = -5(x^2+y^2)^{-2} \cdot 2x = \frac{2}{5}$

$f_y(-1,2) = \frac{\partial}{\partial y}\left[\frac{5}{x^2+y^2}\right] = -5(x^2+y^2)^{-2} \cdot 2y = -\frac{4}{5}$

$f(-1,2) = 3$

$\Rightarrow L(x,y) = 3 + \frac{2}{5}(x+1) + (-\frac{4}{5})(y-2)$

$\Rightarrow \boxed{L(x,y) = \frac{2}{5}x - \frac{4}{5}y + 5}$

5. Find linear approximation to $f(x,y) = xe^{xy}$ at $(1,0,1)$

Solution

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$f_x(1,0) = e^{xy}(1+xy) = 1$

Note: apply chain Rule

$\frac{\partial}{\partial x}[xe^{xy}] = 1 \cdot e^{xy} + x \cdot y \cdot e^{xy}$

$f_y(1,0) = x^2 e^{xy} = 1$

$f(1,0) = 1$

$\Rightarrow L(x,y) = 1 + 1 \cdot (x-1) + 1 \cdot (y-0)$

$\Rightarrow \boxed{L(x,y) = x+y}$

11:26 – 11:38 AM. 12min

6. Find the equation for the tangent planes to the surface

$$z^2 - \frac{x^2}{16} - \frac{y^2}{9} - 1 = 0.$$

at $P_1(4, 3, -\sqrt{3})$ and $P_2(-8, 9, \sqrt{14})$

$\boxed{\text{solution}}$:

To find equation for a plane, we need:

- a point
- the normal vector. (in this case, $\vec{\nabla}F$)

$$\vec{\nabla}F(x,y,z) = \langle -\tfrac{1}{8}x, -\tfrac{2}{9}y, 2z \rangle$$

To get the tangent plane, we use the equation from #1

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = 0$$

Note: this equation is also written as $\vec{\nabla}F \cdot \vec{P_0P}$ where P_0 is the point given, $P(x,y,z)$ is on the plane.
(a,b,c)

▼ plane at $P_1(4, 3, -\sqrt{3})$ is:

$$-\frac{1}{8}(4)(x-4) + (-\frac{2}{9})(3)(y-3) + 2(-\sqrt{3})(z+\sqrt{3}) = 0$$

$$\Rightarrow \boxed{-\frac{1}{2}x - \frac{2}{3}y - 2\sqrt{3}z = 2}$$

▼ plane at $P_2(-8, 9, \sqrt{14})$ is:

$$-\frac{1}{8}(-8)(x+8) - \frac{2}{9}(9)(y-9) + 2(\sqrt{14})(z-\sqrt{14}) = 0.$$

$$\Rightarrow \boxed{x - 2y + 2\sqrt{14}z = 2}$$

7. Find the points at which the surface

$$x^2+2y^2+z^2-2x-2z=0$$

has a horizontal tangent plane.

$\boxed{\text{Solution}}$:

First we think of what is the graph look like.

Notice we have x^2 and $-2x$ term, that reminds me

$(x-1)^2 = x^2-2x+1$; also recall the equation of ellipsoid

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}+\frac{(z-t)^2}{c^2}=1$$

△ with center (h,k,t)

we want to transform the given surface equation to this form:

$$x^2+2y^2+z^2-2x-2z=0$$

$$(x^2-2x+1)+2y^2+(z^2-2z+1)=2$$

$$(x-1)^2+2y^2+(z-1)^2=2$$

$$\Rightarrow \frac{(x-1)^2}{2}+y^2+\frac{(z-1)^2}{2}=1$$

this is a ellipsoid center at $(1,0,1)$, $a=\sqrt{2}$, $b=1$, $c=\sqrt{2}$.

Since horizontal plane has normal vector $\parallel \vec{j}\ (0,0,1)$

let's set normal vector $\vec{n}=\langle 0,0,k\rangle$

Recall the $\vec{\nabla} F$ is $\vec{n}$ (as we proved in #1)

$$\vec{\nabla} F=\langle 2x-2,\ 4y,\ 2z-2\rangle$$

$$\Rightarrow \begin{cases} 2x-2=0 \\ 4y=0 \end{cases}$$

$\Rightarrow x=1,\ y=0.$

<Not yet>

☺ trying to figure out plot @ Matlab

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2/15 7:31 PM – 8:57 PM
1h26min

we can find the point with $x=1$, $y=0$ by plug x, y into surface:

$$1+0+z^2-2(1)-2z=0.$$

$$\Rightarrow \quad z^2-2z-1=0$$

$$\Rightarrow \quad z=1\pm\sqrt{2}$$

so the two points $(1, 0, 1+\sqrt{2})$, $(1, 0, 1-\sqrt{2})$

has a horizontal tangent plane.

(Because at these two points $\vec{\nabla}F$, which is normal vector to the tangent plane, is $\langle 0, 0, k \rangle$)

2115 time include in P8.