Lesson 11 Activities.

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1. Calculate and interpret the directional derivative

A. Derive the limit definition for directional derivative.
Make explicit connection between this limit definition and the limit definition of ordinary derivative.
Explain how limit is used to transform secant to tangent.

Solution:

Let's define surface $z=f(x,y)$, we want to define limit definition of directional derivative at point (a,b) in the direction of unit vector $\vec{u}=\langle u_1,u_2\rangle$.

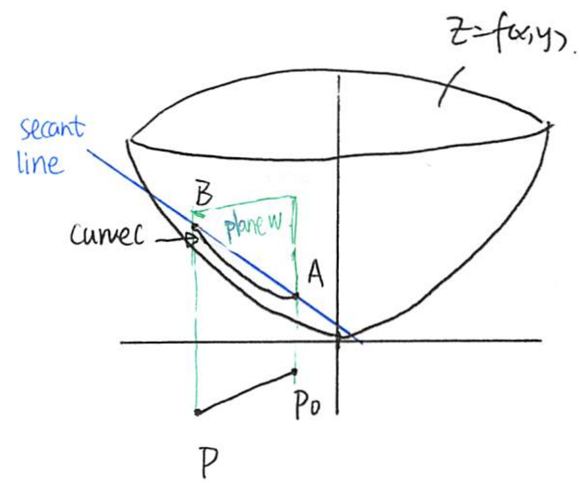

Figure 11.1.1

① use $\vec{l}(h)=\langle a,b\rangle+h\langle u_1,u_2\rangle$ in domain $\mathbb{R}^2$
this line defines input
$P_0(a,b)$ $P(a+hu_1, b+hu_2)$.

② Project $\vec{l}(h)$ onto surface $z=f(x,y)$, to trace the curve C

$$z(h)=f(\vec{l}(h))=f(x(h),y(h))$$
$$=f(a+hu_1, b+hu_2).$$

Curve $C=\{x(h), y(h), f(a+hu_1, b+hu_2)\}$.

$A(x(0), y(0), z(0)) = A(a,b,f(a,b))$
$B(x(h), y(h), z(h)) = B(a+hu_1, b+hu_2, f(a+hu_1, b+hu_2))$

2/1 8:30AM - 8:50AM 20min.

③ slope of the secant line thru poin A & B

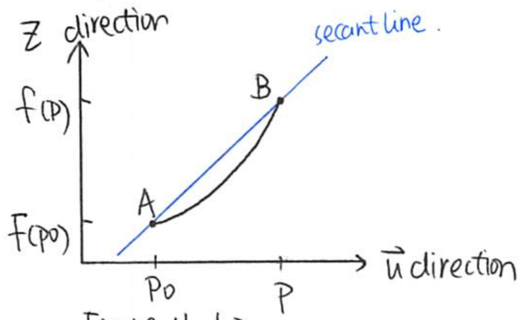

Figure 11.1.2

visualize perpendicular to the plane w

$$m_{AB} = \frac{f(P) - f(P_0)}{P - P_0} = \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

Recall: $P(a+hu_1, b+hu_2)$

$P_0(a,b)$

$\vec{PP_0} = \langle hu_1, hu_2\rangle = h\cdot\langle u_1, u_2\rangle$

$\Rightarrow \|\vec{PP_0}\|_2 = |h| \cdot \sqrt{u_1^2+u_2^2} = |h|$

b/c $\langle u_1, u_2\rangle$ is a normal vector, $\sqrt{u_1^2+u_2^2} = 1$.

④ to find the tangent line slope.

use limit of the secant line, let P move to P_0,

$$\lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h} = D_{\vec{u}}\, f(a,b)$$

This is the directional derivative of f at (a,b) in the direction of $\vec{u}$.

Connection between this limit definition and limit definition of ordinary derivative:

- we use a unit vector $\vec{u}$ to define the direction, and transform the directional derivation problem in 3D into an ordinary derivative in 2D.
- Both of these limits used same idea: find the slope of a secant line, then let the two points gets closer, (using limit) to find the slope of the tangent line at a specific point.

2/5 10:27AM − 10:47 20min.

2. The gradient and level curves

The level curves to the surface $z = x^2+y^2$ are circles in $\mathbb{R}^2$ centered at the origin. Consider input points:

A: P(3, -2)   B: (-2, -1)   C: (-5, 12)

Use this explicit equation for the elliptic paraboloid and the given points to:

(i) Graph the contour curve and level curve associated with each input point.

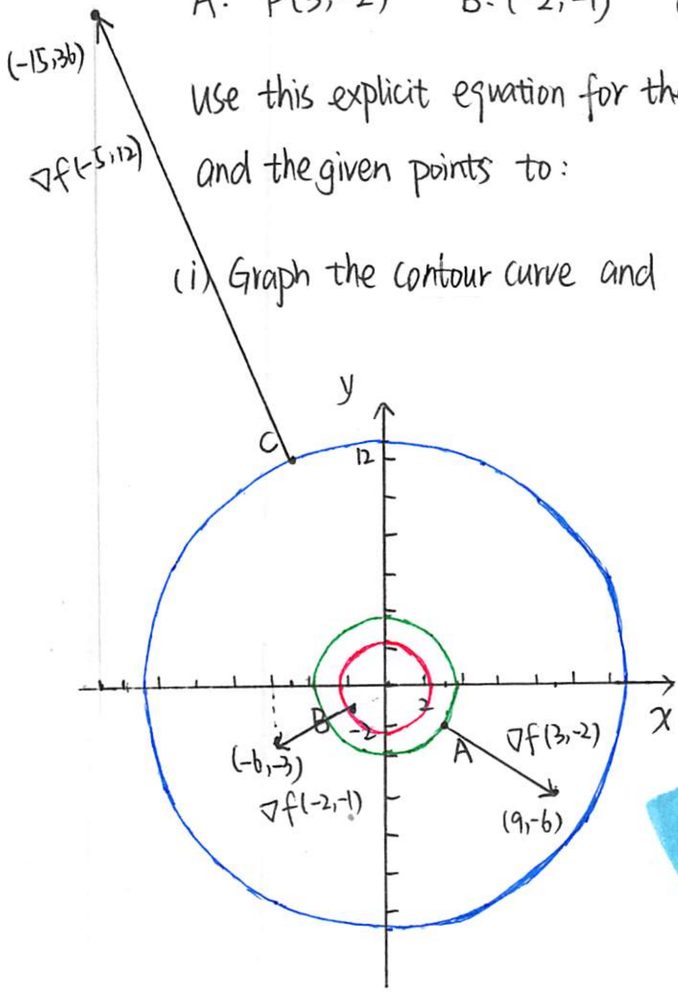

Figure 11.2.1 level curve

Note: these three vectors at A, B, C are the graph for part (iii)

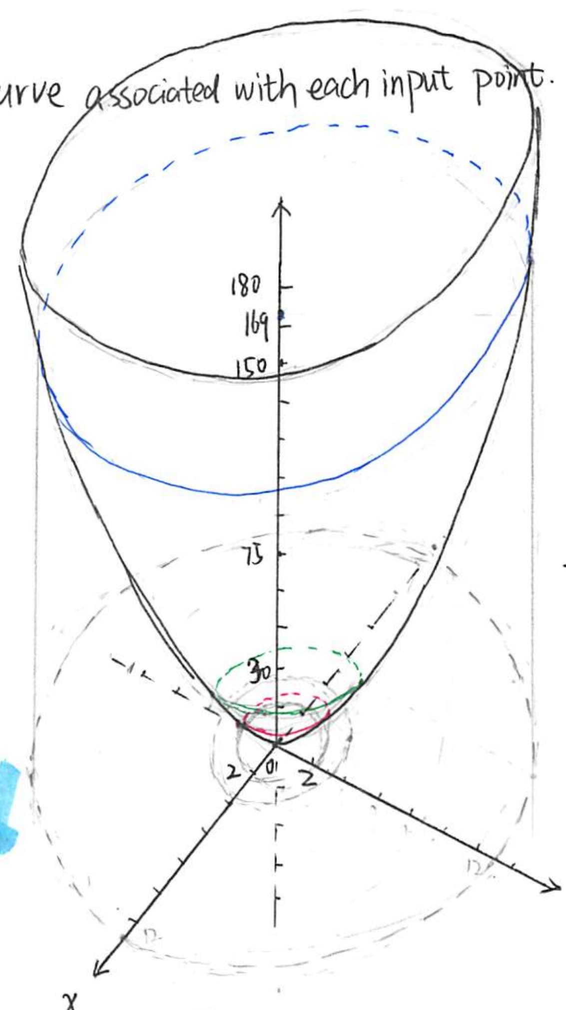

Figure 11.2.2 Contour curve

x	y	z
3	-2	13
-2	-1	5
-5	12	169

1h 40min   2/5   10:55 AM – 12:15 PM
14:30 PM – 15:50 PM

(ii) Determine gradient vector at each input point

solution: Let $f(x,y) = z = x^2 + y^2$

$\nabla f(x,y) = \langle f_x(x,y),\ f_y(x,y) \rangle$ Gradient vector definition.

$= \langle 2x,\ 2y \rangle$

A. $P(3,-2)$.

$\nabla f(3,-2) = \langle 6, -4 \rangle$

B. $P(-2,-1)$

$\nabla f(-2,-1) = \langle -4, -2 \rangle$

C. $P(-5,12)$

$\nabla f(-5,12) = \langle -10, 24 \rangle$

(iii) Graph the gradient vector on level curve, assuming the tail of each vector is the given point P.

Solution: Graph in Figure 11.2.1

Note: the question ask to use P as the tail, so we need to add x, y value of the gradient vector and x, y value of P
(since by convention, vector start at origin).

$\nabla f(3,-2) = \langle 6, -4 \rangle$ tail $\langle 3, -2 \rangle$ head $\langle 6+3, -4-2 \rangle = \langle 9, -6 \rangle$

$\nabla f(-2,-1) = \langle -4, -2 \rangle$ tail $\langle -4, -2 \rangle$ head $\langle -4-2, -2-1 \rangle = \langle -6, -3 \rangle$

$\nabla f(-5,12) = \langle -10, 24 \rangle$ tail $\langle -10, 24 \rangle$ head $\langle -10-5, 24+12 \rangle = \langle -15, 36 \rangle$.

continue #2 ...

(iv) Find a parametric equation for the tangent line to the level curve at each point P.

Solution:

Recall the equation of a line in parametric form is given by:

$$\vec{r}(t) = \vec{r_0} + t \cdot \vec{v}$$

where $\vec{r_0}$ is the point on the line, $\vec{v}$ gives the direction.

① $\vec{r_0}$ here is the point P, since the point P is on the "tangent line at P".

② $\vec{v}$ is orthogonal to the gradient vector we found in (ii)

▼ tangent line at A: P(3,−2)

since $\nabla f(3,-2) = \langle 6,-4 \rangle$, slope of gradient vector at A is: $m_{\nabla f} = \frac{-4}{6} = \frac{-2}{3}$,

$\Rightarrow$ tangent line $\perp$ gradient vector, $m_{tan} \cdot m_{\nabla f} = -1$

$\Rightarrow m_{tan} = \frac{3}{2}$,

$\Rightarrow \vec{V_A} = \langle 2,3 \rangle$

So tangent line at P(3,−2) is:

$$\vec{r}(t_A) = \langle 3,-2 \rangle + t\langle 2,3 \rangle = \langle 3+2t, -2+3t \rangle$$

▼ tangent line at B: P(−2,−1)

Since $\nabla f(-2,-1) = \langle -4,-2 \rangle \Rightarrow \vec{V_B} = \langle -1,2 \rangle$

Note: same reason as A

So tangent line at P(−2,−1) is:

$$\vec{r}(t_B) = \langle -2,-1 \rangle + t\langle -1,2 \rangle = \langle -2-t, -1+2t \rangle$$

▼ tangent line at C: P(−5,12)

Since $\nabla f(-5,12) = \langle -10,24 \rangle \Rightarrow \vec{V_C} = \langle 12,5 \rangle$

So tangent line at P(−5,12) is:

$$\vec{r}(t_C) = \langle -5,12 \rangle + t\langle 12,5 \rangle = \langle -5+12t, 12+5t \rangle$$

2/5 14:25 − 14:30 27min
15:00 − 16:22

3. The Gradient and Steepest Descent

Consider the paraboloid $f(x,y) = 16 - \frac{x^2}{4} - \frac{y^2}{16}$

P is the point on a level curve of $f(x,y)$

Compute the slope of the line tangent to level curve at P.

Verify the tangent line is orthogonal to gradient at P. (Next page).

A. Level curve $f(x,y) = 0$, $P(0,16)$

$$f(x,y) = 16 - \frac{x^2}{4} - \frac{y^2}{16} = 0$$

$$\Rightarrow \frac{x^2}{4} + \frac{y^2}{16} = 16$$

$$\Rightarrow \frac{x^2}{64} + \frac{y^2}{256} = 1$$

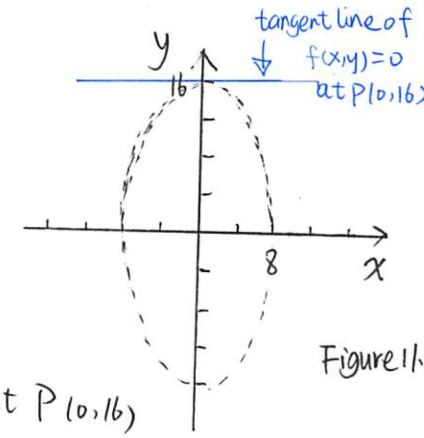

Figure 11.3.1

From the graph we can tell the slope at $P(0,16)$ is $m_P = 0$ (horizontal line). ~~⇒ tangent line:~~

B. level curve $f(x,y) = 12$, $P(4,0)$

$$f(x,y) = 16 - \frac{x^2}{4} - \frac{y^2}{16} = 12$$

$$\Rightarrow \frac{x^2}{4} + \frac{y^2}{16} = 4$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{64} = 1$$

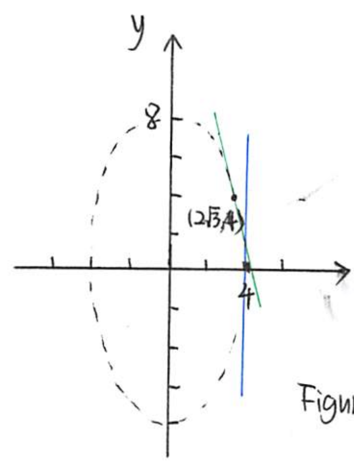

Figure 11.3.2

(blue line Figure 11.3.2)

From the graph we know slope of tangent line at $P(4,0)$ Does Not exist

(Reason: vertical line, slope undefined) Q: can we say the slope is $\pm\infty$?

C. level curve $f(x,y) = 12$, $P(2\sqrt{3}, 4)$

Note: This is the same level curve in part B. So the graph of level curve is the same.

tangent line: (green line Figure 11.3.2)

Let's rephrase this question as:

Given $\frac{x^2}{16} + \frac{y^2}{64} = 1$, find the slope at $P(2\sqrt{3}, 4)$

Recall slope of tangent line is same as $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(\frac{x^2}{16} + \frac{y^2}{64}\right) = \frac{d}{dx}(1)$$ apply chainRule

$$\frac{x}{8} + \frac{y}{32} \cdot \boxed{\frac{dy}{dx}} = 0$$

$$\frac{dy}{dx} = -\frac{4x}{y}$$

$\Rightarrow$ So the slope at $P(2\sqrt{3}, 4)$ is: Point

$$\frac{dy}{dx} = -\frac{4 \times 2\sqrt{3}}{4} = -2\sqrt{3}$$ slope

~~So the~~

verify the tangent line $\perp$ gradient at P:

$\nabla f = \langle f_x(x,y), f_y(x,y) \rangle = \langle -\frac{1}{2}x, -\frac{1}{8}y \rangle$

Q: Is this the correct?

A. at $P(0,16)$ $\nabla f = \langle -\frac{1}{2} \times 0, -\frac{1}{8} \times 16 \rangle = \langle 0, -2 \rangle$

The slope of the line along ∇f is $m_{\nabla f} = \frac{-2}{0}$ (Does Not defined)

So the tangent line orthogonal to gradient at $(0,16)$

B. at $P(4,0)$ $\nabla f = \langle -\frac{1}{2} \times 4, -\frac{1}{8} \times 0 \rangle = \langle -2, 0 \rangle$

$m_{\nabla f} = \frac{0}{-2} = 0$. since the slope of tangent line is undefined.

we say the tangent line orthogonal to gradient at $(4,0)$

C. at $P(2\sqrt{3}, 4)$ $\nabla f = \langle -\frac{1}{2} \times 2\sqrt{3}, -\frac{1}{8} \times 4 \rangle = \langle -\sqrt{3}, -\frac{1}{2} \rangle$

$m_{\nabla f} = \frac{1}{2\sqrt{3}}$ Recall the slope of tangent line at $(4,0)$ is $-2\sqrt{3}$

$\frac{1}{2\sqrt{3}} \times -2\sqrt{3} = -1$, so we say the gradient orthogonal to tangent line at $(4,0)$

Note:

- a line with slope 0 is parallel to x axis, a line with undefined slope parallel to y axis
- if $m_{l_1} \times m_{l_2} = -1$, $l_1 \perp l_2$.

2/6 17:37–18:13 35min