Math 1C, Part I: Multivariable Differentiation

Lesson 6: Planes and Surfaces

Recall that in lesson 5 we studied vector-valued functions of the for

$\vec{r}(t) = \langle x(t), y(t) \rangle$ or $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\vec{r}: \underbrace{D}_{\text{Domain}} \longrightarrow \underbrace{\mathbb{R}^2}_{\text{Codomain}}$ for $D \subseteq \mathbb{R}$ $\quad$ $\vec{r}: \underbrace{D}_{\text{Domain}} \longrightarrow \underbrace{\mathbb{R}^3}_{\text{Codomain}}$ for $D \subseteq \mathbb{R}$

These functions had a single input variable (since Domain $D \subseteq \mathbb{R}$) and two or three output variables (since the codomain was either $\mathbb{R}^2$ or $\mathbb{R}^3$). Moreover, we used these functions to create lines or curves in $\mathbb{R}^2$ or $\mathbb{R}^3$.

In this Lesson 6, we will begin our study of surfaces in $\mathbb{R}^3$.

When considering planes and surfaces in $\mathbb{R}^3$,

we will categorize these graphs into two groups.

To understand this, let's go back to our knowledge

of lines and curves in $\mathbb{R}^2$.

isolated output variable

written explicitly in terms of a single input variable

1. Explicit functions in the form $y = f(x)$

have one input variable and for each input

we have a only one output. When we produce

a graph of these functions, the graphs

pass the vertical line test

↳ To graph these in Mma, we use Plot[ ] command.

2. Implicit relations in the form $F(x,y) = 0$

are given as equations in two variables

and describe curves in $\mathbb{R}^2$ that don't

necessarily pass the vertical line test.

↳ To graph these in Mma, we use the ContourPlot[ ] command

When producing graphs of planes and surfaces in $\mathbb{R}^3$ we will use the same categories in a more general context:

1. Explicit functions in the form

$$\underbrace{z}_{\substack{\text{isolated output} \\ \text{variable}}} = \underbrace{f(x, y)}_{\substack{\text{written explicitly in terms} \\ \text{of two output variables}}}$$

result in a set ordered triplets (x, y, z) that can be graphed on the xyz axes. In this case, each unique input ordered pair (x, y) produces only one output z. The graphs of these functions $f: D \longrightarrow \mathbb{R}$ where $D \subseteq \mathbb{R}^2$ pass the vertical line test in $\mathbb{R}^3$.

To graph these in Mma, we use the Plot3D[ ] command

2. Implicit relations in the form

$$\underbrace{F(x, y, z)}_{\text{an expression in three variables in which it is "hard" to isolate } z \text{ in terms of } x \text{ and } y} = 0$$

equation in 3 variables that describes a surface in $\mathbb{R}^3$

(hard may mean that it takes lots of work, it is impossible or that by doing so, we lose information).

In this general form, we will set the RHS equal to zero.

result in ordered triplets (x, y, z) that satisfy our equation. However, each set of (x, y) does not necessarily correspond with a unique z value.

To graph these in Mma, we use the ContourPlot3D[ ] command

Equations for a Plane in $\mathbb{R}^3$

Let's begin our discussion of planes in $\mathbb{R}^3$ by revisiting our work with lines in $\mathbb{R}^2$. So far we have two techniques to graph lines in $\mathbb{R}^2$:

A. Explicit Scalar-valued functions:

Point-Slope form: $y = f(x) = m(x - x_0) + y_0$

point on line: (x_0, y_0)
slope of line: m.

Slope intercept form: $y = f(x) = mx + b$

y-intercept of line: $(0, b)$
Slope of line: m

B. Explicit vector-valued function

$\vec{r}(t) = \vec{r}_0 + t \cdot \vec{v}$

point on line: $\vec{r}_0 = \langle x_0, y_0 \rangle$
direction of line: $\vec{v} = \langle a, b \rangle$
(corresponding to slope $m = \frac{b}{a}$)

$$= \langle x_0, y_0 \rangle + t \cdot \langle a, b \rangle$$

$$= \langle \underbrace{x_0 + at}_{x(t)}, \underbrace{y_0 + bt}_{y(t)} \rangle$$

Both of these techiques encode a line explicitly in terms of it's slope $m = \frac{b}{a}$ (or direction $\vec{v} = \langle a, b \rangle$)

However, the is a third technique that does not explicitly reference the slope but instead implicitly encodes this information with a clever mapping.

Example: Recall our example from lesson 5 of a line through the points $(\underset{x_0}{0}, \underset{y_0}{-2})$ and $(\underset{x_1}{2}, \underset{y_1}{-3})$. We saw the two explicit representations:

A. Explicit scalar-valued function

We calculate slope $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-3-(-2)}{2-0} = -\dfrac{1}{2}$

and find the point slope form

$$y = f(x) = -\frac{1}{2}(x-0) - 2 = -\frac{1}{2}x - 2$$

B. Explicit vector-valued function

We have point $\vec{r}_0 = \langle x_0, y_0 \rangle = \langle 0, -2 \rangle$ and

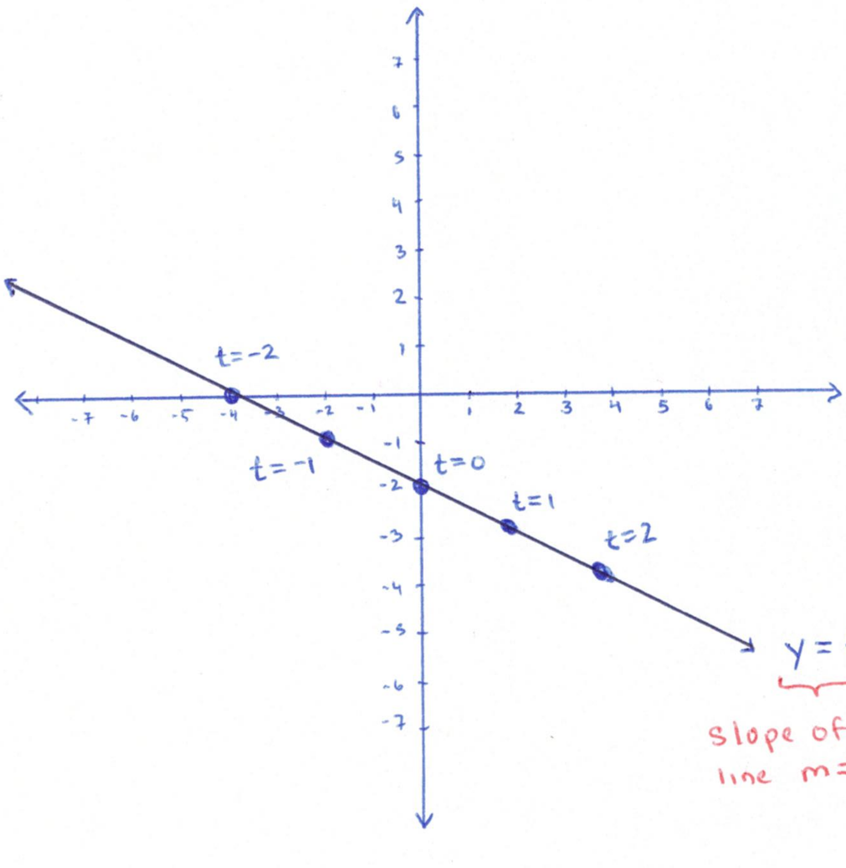
direction $\vec{v} = \langle a, b \rangle = \langle 2, -1 \rangle$ (since $m = \frac{b}{a} = \frac{-1}{2}$

$$\Rightarrow \quad \vec{r}(t) = \vec{r}_0 + t \cdot \vec{v}$$

$$= \langle 0, -2 \rangle + t \langle 2, -1 \rangle$$

$$= \langle \underbrace{2t}_{x(t)} , \underbrace{-2 - t}_{y(t)} \rangle$$

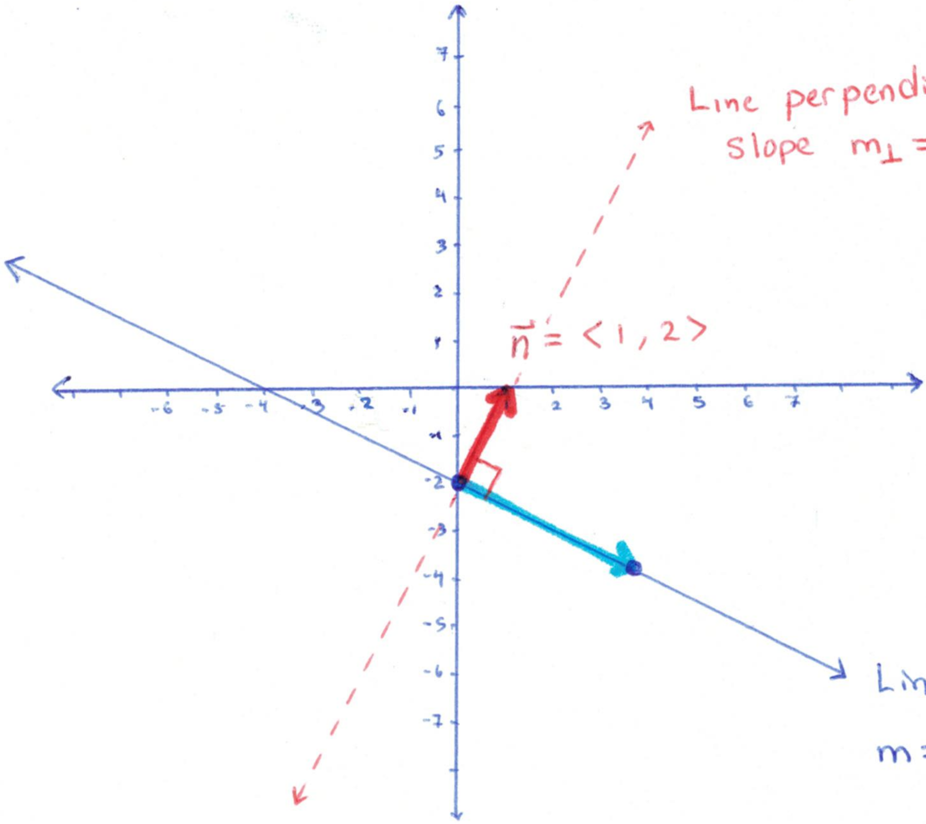
Now, let's graph this line in $\mathbb{R}^2$



$\underbrace{y = f(x)}_{\text{slope of line } m = \frac{-1}{2}}$ or $\underbrace{\vec{r}(t) = \vec{r}_0 + t \cdot \vec{v}}_{\text{direction of line } \vec{v} = \langle 2, -1 \rangle}$

L6, p7

Notice, we could also encode this line as all the points P in $\mathbb{R}^2$ with $P(x,y)$ such that vector $\vec{x} = \overrightarrow{P_0P}$ from $P_0(x_0, y_0)$ to $P(x,y)$ in Orthogonal (perpendicular) to normal vector $\vec{n} = \langle 1, 2 \rangle$



$L = \{ \vec{x} \in \mathbb{R}^2 : \vec{n} \cdot \vec{x} = 0$ where $\vec{n} = \langle 1, 2 \rangle$ and $\vec{x} = \langle x - 0, y - (-2) \rangle \}$

$\Rightarrow L = \{ \langle x, y \rangle : \langle 1, 2 \rangle \cdot \langle x - 0, y + 2 \rangle = 0 \}$

$\Rightarrow L = \{ \langle x, y \rangle : 1(x-0) + 2(y+2) = 0 \}$

Then the implicit relation given by equation

$$x + 2y + 4 = 0$$

encodes this line. Notice, the "slope" of the line is not explicitly written but instead is implicitly encoded in normal vector $\vec{n} = \langle 1, 2 \rangle$.
(each slope corresponds to a unique orthogonal direction)

We can generalize from this example to propose a third techique to define a line. A line L with slope $m = \frac{b}{a}$ (corresponding to direction $\vec{v} = \langle a, b \rangle$) through point $P_0(x_0, y_0)$ is the set of all vectors $\vec{x} = \overrightarrow{P_0P}$ with initial point P_0 and terminal point $P(x, y)$ such that $\vec{x} \perp \vec{n}$ where $\vec{n} = \langle b, -a \rangle$ is the normal vector to $\vec{v}$

$\boxed{\begin{array}{l} \text{note: } \vec{n} \cdot \vec{v} = 0 \\ \Leftrightarrow \vec{n} \perp \vec{v} \end{array}}$

$$\Rightarrow \quad \vec{n} \cdot \vec{x} = 0$$

$$\Rightarrow \quad \langle b, -a \rangle \cdot \langle x - x_0, y - y_0 \rangle = 0$$

$$\Rightarrow \quad b \cdot (x - x_0) + -a(y - y_0) = 0$$

Equations of Planes

We can use this intuition for equations of lines in $\mathbb{R}^2$ to generalize and create equations for planes in $\mathbb{R}^3$.

We might think of a plane in $\mathbb{R}^3$ in a few different ways:

A. The span of two vectors in different directions

Suppose $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $\vec{x}$ is not in same direction of $\vec{y}$ (i.e. $\vec{x} \neq \alpha \vec{y}$ for any $\alpha \in \mathbb{R}$). Then, if we take any possible scalar combination of $\vec{x}$ and $\vec{y}$

$$s \cdot \vec{x} + t \vec{y} + \vec{r_0}$$

$\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ is a "point" on the plane

for all $s, t \in \mathbb{R}$, we would form a flat surface in $\mathbb{R}^3$ known as a plane.

This gives us a vector-valued function for the plane with two input variables and three output variables

$$\vec{r}(s,t) = s \cdot \vec{x} + t \cdot \vec{y} + r_0$$

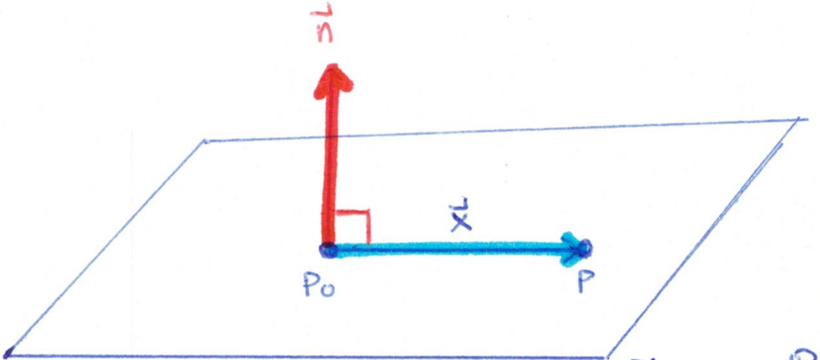
$$= \langle \underbrace{sx_1 + tx_2 + x_0}_{x(s,t)}, \underbrace{sy_1 + ty_2 + y_0}_{y(s,t)}, sz_1 + tz_2 + z_0 \rangle$$

We might also recognize that a plane is uniquely determined by a point on the plane $P_0(x_0, y_0, z_0)$ and a nonzero vector ortnogonal to the plane $\vec{n} = \langle a, b, c \rangle$

(this will be called the normal vector to the plane)

In this way, we define a plane with the following diagram in mind:

Any set of two vectors that are not colinear correspond to a unique normal direction



Plane in $\mathbb{R}^3$:

- Specific point on Plane: $P_0(x_0, y_0, z_0)$
- General point on Plane: $P(x, y, z)$
- Normal vector to Plane: $\vec{n} = \langle a, b, c \rangle$

Given a specific point on the plane $P_0(x_0, y_0, z_0)$ and normal vector $\vec{n} = \langle a, b, c \rangle$, the plane is the set of points $P(x, y, z) \in \mathbb{R}^3$ for which vector $\vec{x} = \overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$ is ortnogonal to $\vec{n}$.

This leads to two equivalent techniques to implicitly define the plane in terms of its normal vector

Vector Equation for Plane:

Given a point on plane $P_0(x_0, y_0, z_0)$

and a normal vector to plane $\vec{n} = \langle a, b, c \rangle$,

the plane is a collection of all points

$P(x, y, z) \in \mathbb{R}^3$ so that vector $\vec{x} = \overrightarrow{P_0P}$

is orthogonal to $\vec{n}$ with

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \overrightarrow{P_0P} = 0$$

Scalar Equation for plane

Using the assumptions above, we see our plane is implicitly defined by equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- The coefficients $a, b, c \in \mathbb{R}$ determine the orientation of the plane.
- Notice, the equation of plane depends only on the direction of normal vector $\vec{n}$ not on the magnitude of this vector.

Then, we can use this intuition and our Knowledge of dot products to generate an implicit relation via a scalar equation for the plane: To do so, we want to find all points $P(x,y,z)$ such that for $\vec{x} = \overrightarrow{P_0P}$ we have

$$\vec{n} \perp \vec{x} \Rightarrow \vec{n} \cdot \vec{x} = 0$$

$$\Rightarrow \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\Rightarrow \boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0}$$

$$\Rightarrow ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$\Rightarrow ax + by + cz = ax_0 + by_0 + c \cdot z_0$$

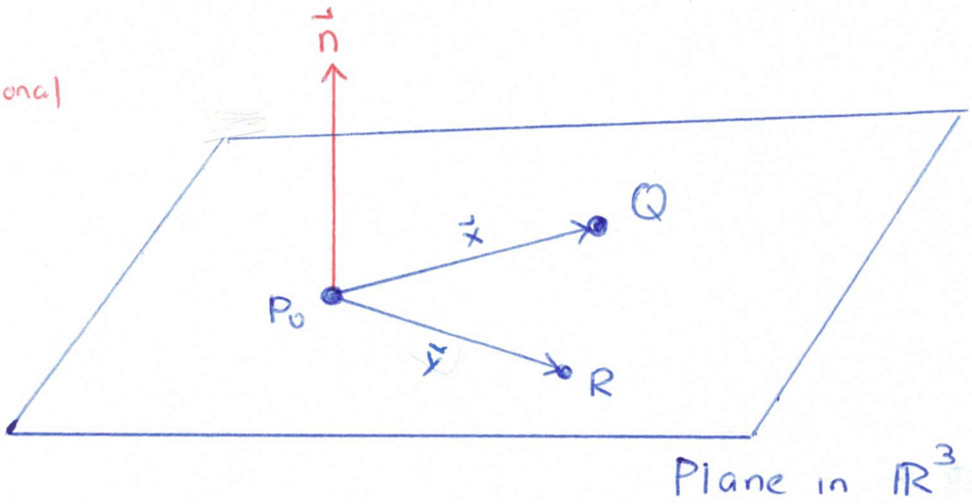
$$\Rightarrow \boxed{ax + by + cz = d \quad \text{where } d = ax_0 + by_0 + cz_0}$$

Example 12.1.2 p. 859)

Find the plane that passes through the (non colinear) points $P_0(2,-1,3)$, $Q(1,4,0)$ and $R(0,-1,5)$

Solution: Let's visualize this problem. We are told that we have three points on a plane:

WTF $\vec{n}$ orthogonal to $\vec{x}$ and $\vec{y}$.



To create an equation for a plane we need a point on the plane, say $P_0(2,-1,3)$ and a normal vector to plane. To find our normal vector, we note that since points P, Q, R are on plane, we must have that vectors $\vec{x} = \vec{PQ}$ and $\vec{y} = \vec{PR}$ are also on plane (as seen above).

Then for $\vec{x} = \overrightarrow{PR} = \langle 1-2, 4+1, 0-3 \rangle$

$$= \langle -1, 5, -3 \rangle$$

and $\vec{y} = \overrightarrow{PQ} = \langle 0-2, -1+1, 5-3 \rangle$

$$= \langle -2, 0, 2 \rangle$$

We see the vector

$$\vec{n} = \vec{x} \times \vec{y}$$

$$= \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ -1 & 5 & -3 \\ -2 & 0 & 2 \end{array} \right| \begin{array}{cc} \vec{i} & \vec{j} \\ -1 & 5 \\ -2 & 0 \end{array}$$

$$= 10\vec{i} + 6\vec{j} + 0\vec{k}$$
$$+ 0\vec{i} + 2\vec{j} + 10\vec{k}$$

$\Rightarrow \vec{n} = \langle 10, 8, 10 \rangle$ is one possible normal vector

Then, we can define a plane as the set of points $P(x, y, z) \in \mathbb{R}^3$ such that vector $\vec{x} = \overrightarrow{P_0 P}$ is orthogonal to $\vec{n}$:

$$\Rightarrow \quad \vec{n} \cdot \vec{x} = 0$$

$$\Rightarrow \quad \langle 10, 8, 10 \rangle \cdot \langle x - 2, \; y - (-1), \; z - 3 \rangle = 0$$

$$\Rightarrow \quad 10(x-2) + 8(y+1) + 10(z-3) = 0$$

$$\Rightarrow \quad 10x + 8y + 10z - 20 + 8 - 30 = 0$$

$$\Rightarrow \quad 10x + 8y + 10z = 42$$

$$\Rightarrow \quad 5x + 4y + 5z = 21$$

Notice: We see in the example above that any scalar multiple of $\vec{n}$ can be used to define plane. We could have started with

$$\vec{n}_1 = \frac{1}{2}\vec{n} = \langle 5, 4, 5 \rangle$$

and ended with same equation

(i.e. the direction, not the magnitude, of the normal determines plane)

Definition: Parallel and orthogonal Planes

- Two distinct planes are parallel if their respective normal vectors are parallel. In other words, if the normal vectors of two different planes are in the same direction (one is a scalar multiple of the other), we say the planes are parallel.

- Two planes are orthogonal if their normal vectors are orthogonal (i.e. if the dot product between the normal vectors is equal zero).

Example 12.1.4 p. 861)

Determine which of the following planes are parallel and which are orthogonal:

Plane 1: $2x - 3y + 6z = 12$

Plane 2: $-x + \frac{3}{2}y - 3z = 14$

Plane 3: $6x + 8y + 2z = 1$

Plane 4: $-9x - 12y - 3z = 7$

Solution: Let $\vec{n}_i$ be the normal vector to plane i for $i = 1, 2, 3, 4$. Then, since we can read the components of the normal vector from the scalar coefficients of $x, y,$ and z with

$$ax + by + cz = d \iff \vec{n} = \langle a, b, c \rangle$$

we can write each $\vec{n}_i$ as follows

$\vec{n}_1 = \langle 2, -3, 6 \rangle$

$\vec{n}_2 = \langle -1, \frac{3}{2}, -3 \rangle$

$\vec{n}_3 = \langle 6, 8, 2 \rangle$

$\vec{n}_4 = \langle -9, -12, -3 \rangle$

Notice that we have two relations

$$\vec{n}_1 = -2 \cdot \vec{n}_2 \quad \text{and} \quad \vec{n}_4 = -\frac{3}{2} \cdot \vec{n}_3$$

Then, by definition, we know plane 1 and plane 2 are parallel since their normal vectors in same direction. Similarly, plane 3 and 4 are parallel. Further, we notice

$$\vec{n}_1 \cdot \vec{n}_3 = 0 \quad \text{and} \quad \vec{n}_1 \cdot \vec{n}_4 = 0$$

$\Rightarrow$ plane 1 is ortnogonal to plane 3 and 4.

$\Rightarrow$ plane 2 is ortnogonal to plane 3 and 4.

$\boxed{\text{L6,P}}$

Let's confirm graphically using Mm

Example 12.1.6 p. 861)

Find the equation of the line of intersection between the two planes given be equations

$$\underbrace{x + 2y + z = 5}_{\text{implicit equation of plane 1}} \quad \& \quad \underbrace{2x + y - z = 7}_{\text{implicit equation of plane 2}}$$

Solution: First, we note the normal vectors to each plane are given by

$$\vec{n}_1 = \langle 1, 2, 1 \rangle \quad \text{and} \quad \vec{n}_2 = \langle 2, 1, -1 \rangle$$

We see $\vec{n}_1 \neq \alpha \cdot \vec{n}_2$ for all $\alpha \in \mathbb{R}$.

(these vectors are not scalar multiples of each other)

Thus, plane 1 and plane 2 are not parallel and intersect in a line: call it $\vec{r}(t)$.

To find the vector-valued equation for

$$\vec{r}(t) = \vec{r}_0 + t \cdot \vec{v}$$

we need two pieces of information:

$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ a "point" on the line

$\vec{v} = \langle a, b, c \rangle$ the direction of line.

Step 1: Find point on line

To find a point $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ on line $\vec{r}(t)$ assume (for simplicity) $z_0 = 0$.

Then, we want to find $x_0, y_0 \in \mathbb{R}$ such that both equations for the planes are true:

Equaton 1: $x_0 + 2y_0 = 5$

Equaton 2: $2x_0 + y_0 = 7$

From equation 1, we see

$$x_0 = 5 - 2y_0$$

Substituting this back into equation 2, we see

$$2 \cdot (5 - 2y_0) + y_0 = 7$$

$$\Rightarrow \quad 10 - 4y_0 + y_0 = 7$$

$$\Rightarrow \quad -3y_0 = -3$$

$$\Rightarrow \quad y_0 = 1$$

$$\Rightarrow \quad \underbrace{x_0 = 5 - 2 \cdot}_{\text{by equivalent equation to equation 1}} 1 = 3$$

$\Rightarrow$ Point $r_0 = \langle 3, 1, 0 \rangle$ is on plane 1 & 2

Step 2: Find the direction of $\vec{r}(t)$

Notice the line $\vec{r}(t)$ lies in both plane 1 and in plane 2. Since any line on a plane is orthogonal to the normal vector of that plane, we know

Line $\vec{r}(t)$ is orthogonal to $\vec{n}_1$ and $\vec{n}_2$.

Thus, the directon $\vec{v}$ of $\vec{r}(t)$ is parallel to the cross product of $\vec{n}_1$ and $\vec{n}_2$.

So, we might set

$$\vec{v} = \vec{n}_1 \times \vec{n}_2$$

$$= \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{array} \right| \begin{array}{cc} \vec{i} & \vec{j} \\ 1 & 2 \\ 2 & 1 \end{array}$$

$$= -2\vec{i} + 2\vec{j} + \vec{k}$$
$$\quad - \vec{i} + \vec{j} - 4\vec{k}$$

$$= -3\vec{i} + 3\vec{j} - 3\vec{k}$$

$$\Rightarrow \vec{v} = \langle -3, 3, -3 \rangle$$

Then we have our vector-valued equation for our line of intersection given by

$$\vec{r}(t) = \vec{r_0} + t \cdot \vec{v}$$

$$= \langle 3, 1, 0 \rangle + t \cdot \langle -3, 3, -3 \rangle$$

$$= \langle 3 - 3t, \ 1 + 3t, \ -3t \rangle$$

To get the entire line (rather than a line segment), we assume $-\infty < t < \infty$.

Now that we solved this by hand, let's check our work using Mma.

A quadratic surface is described by a quadratic

[cuadro = square
español english]

equation in three variables. Such implicit equations have the general form: $F(x,y,z) = 0$ with

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

The coefficients $A, B, C, \dots, J \in \mathbb{R}$ and at least one of the coefficients A, B, C, D, E, F is non zero.

We will use Mathematica software to graph such surfaces in $\mathbb{R}^3$ (with either Plot3D[ ] or ContourPlot3D[ ])

$\uparrow$ explicit: $z = f(x,y)$ $\qquad$ $\uparrow$ implicit: $F(x,y,z)=0$

When graphing such surfaces, we will use the following tools:

1. Intercepts: Determine the points where the surface intersects the coordinate axis. To find these points, set x, y, z equal to zero in pairs and solve for third variable

2. Traces: Find traces of the surface to visualize cross sections of surface in each direction. For example, setting $z = z_0$ (any constant) gives traces parallel to xy plane.

Example 12.1.8 p. 865) Ellipsoid

Graph the ellipsoid defined by the implicit equation

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} + \frac{z^2}{5^2} - 1 = 0$$

in general form

$F(x, y, z) = 0$

Solution: To create our graph, we will use Mathematica. In particular, since this surface is defined via an implicit relation, we will use the ContourPlot3D[ ] command. However, before we do so, let's do some analysis by hand so we know what to expect.

Step 1: Find coordinate axis intercepts

We begin by finding the coordinate axis intercepts by setting x, y, and z to zero in pairs.

x - intercept (set $y=0$ and $z=0$)

$$\Rightarrow \quad \frac{x^2}{3^2} - 1 = 0$$

$$\Rightarrow \quad \frac{x^2}{3^2} = 1$$

$$\Rightarrow \quad x^2 = 3^2$$

$$\Rightarrow \quad \sqrt{x^2} = \sqrt{9}$$

$$\Rightarrow \quad |x| = 3$$

$$\Rightarrow \quad x = -3 \quad \text{or} \quad x = +3$$

$\Rightarrow$ The ellipsoid has two x-intercepts at points

$(-3, 0, 0)$ and $(3, 0, 0)$ $\leftarrow$ Notice this correspo[nds] to our intuition that the length of the x-semiaxis is $a = 3$

Using similar analysis, we confirm that the two y-intercepts are at $(0, \pm 4, 0)$ and the two z-intercepts occur at points $(0, 0, \pm 5)$.

Step 2: Analyze the traces of this surface.

xy-type traces in "horizontal" planes:

To find the traces in any plane parallel to the xy-plane we set variable $z = z_0$ where z_0 is a constant with $|z_0| \le 5$.

$$\Rightarrow \frac{x^2}{3^2} + \frac{y^2}{4^2} + \frac{z_0^2}{5^2} - 1 = 0$$

$$\Rightarrow \frac{x^2}{3^2} + \frac{y^2}{4^2} = \underbrace{1 - \frac{z_0^2}{5^2}}_{\text{constant number}} = 1 - \frac{z_0^2}{25}$$

This gives equations for a family of "ellipses" in $\mathbb{R}^3$ embedded in the $z = z_0$ plane that "traces" the edge of the surface.

The largest ellipse in this family happens when $z = z_0 = 0$.

This is the xy-trace in the xy plane with the x and y components given by equation

$$\frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$$

in reality, these traces are collections of points in $\mathbb{R}^3$ (we can plot using $\boxed{\text{LG, p 28}}$ ParametricPlot3D) yz - trace in the yz-plane (set $x=0$)

$$\Rightarrow \quad \frac{y^2}{4^2} + \frac{z^2}{5^2} = 1$$

xz - trace in the xz-plane (set $y=0$)

$$\Rightarrow \quad \frac{x^2}{3^2} + \frac{z^2}{5^2} = 1$$

By sketching each of these traces, the wireframe outline of an ellipsoid emerges.

In our discussion of quadratic surfaces, we will focus on the following types:

Implicit: $F(x,y,z)=0$

Ellipsoids: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} + \dfrac{(z-m)^2}{c^2} - 1 = 0$

(3D version of an ellipse)

center at (h, k, m)

length of x-semiaxis : a

length of y-semiaxis : b

length of z-semiaxis : c

Elliptic Paraboloid: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} - z = 0$ ← implicit $F(x,y,z)=0$

$\Rightarrow \quad z = \dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2}$ ← explicit: $z = f(x,y)$

unique vertex at input : (h, k)

Elliptic Cone: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$ ← Implicit: $F(x, y, z) = 0$

$$\Rightarrow \frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Hyperbolic Paraboloid: $$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0$$ ← Implicit: $F(x,y,z)=0$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Example 12.1.9 p. 865-866) Elliptic Paraboloid

Graph the elliptic paraboloid defined by the explicit function definition

$$z = \frac{x^2}{4^2} + \frac{y^2}{2^2}$$

in general form:

$z = f(x,y)$

Since output variable z is written explicitly in terms of x and y we will use Mma's Plot3D[] command

Solution: We will graph this using Mma software. Before we do so, let's do some analysis on paper.

Step 1: Find intercepts with coordinate axis.

x-intercept (set $y=0=z$)

$\Rightarrow \quad 0 = \frac{x^2}{4^2}$

$\Rightarrow \quad x = 0$

$\Rightarrow$ x-intercept is at $(0,0,0)$

With a little thought, we see only one intercept for all coordinate axis.

Step 2A: Find trace(s) in horizontal planes $z = z_0$

Notice for any $z_0 < 0$, we see that

no trace exists since

$$\underset{\substack{\uparrow \\ \text{negative} \\ \text{number}}}{z_0} \neq \underbrace{\frac{x^2}{16} + \frac{y^2}{4}}_{\substack{\text{nonnegative} \\ \text{number}}}$$

Moreover, for $z_0 > 0$, we have

$$z_0 = \frac{x^2}{4^2} + \frac{y^2}{2^2}$$

is an "ellipse" in $\mathbb{R}^3$ embedded in plane $z = z_0$.

Step 2B: Find traces in planes parallel to yz-plane

If we take vertical plane $x = x_0$ $(\vec{n} = \langle 1, 0, 0 \rangle)$

an intersect this with our surface, we get

$$z = \frac{x_0^2}{4^2} + \frac{y^2}{2^2}$$

which is a "parabola" in $\mathbb{R}^3$ embedded in plane $x = x_0$.

Step 2C: Find traces in planes parallel to xz-plane

The trace of the surface in the vertical plane $y = y_0$ is the "parabola" in $\mathbb{R}^3$

$$z = \frac{x^2}{4^2} + \frac{y_0^2}{2^2}$$

embedded in plane $y = y_0$.

Finally, we can get a wireframe outline of the surface by graphing a $z = z_0$ trace, (with $z_0 > 0$) the yz-trace and the xz-trace

$z = 4$ - trace: $\frac{x^2}{4^2} + \frac{y^2}{2^2} = z_0 = 4 = 2^2$

$$\Rightarrow \frac{x^2}{4^2 \cdot 2^2} + \frac{y^2}{2^2 \cdot 2^2} = 1$$

$$\Rightarrow \frac{x^2}{8^2} + \frac{y^2}{4^2} = 1$$

The name elliptic paraboloid represents the idea that the traces of this surface are ellipses and parabol

yz-trace (set $x = 0$): $z = \frac{y^2}{2^2}$

xz-trace (set $y = 0$): $z = \frac{x^2}{4^2}$

Example 12.1.11 p. 867 – 868) A hyperbolic paraboloid

Graph the surface defined by the equation

$$z = \frac{x^2}{1^2} - \frac{y^2}{2^2} = x^2 - \frac{y^2}{4}$$

In general form:

$z = f(x,y)$

Explicit function so we use Plot3D[] command.

Solution: We will graph this surface using Plot3D[] command in Mathematica. To do so, let's do a quick domain and range analysis.

set $y=0 \Rightarrow z = x^2$

$$\Rightarrow \underbrace{x \in \mathbb{R}}_{\text{indicates info about domain}} \quad \& \quad \underbrace{z \geq 0}_{\text{indicates info abut range}}$$

set $x=0 \Rightarrow z = -\frac{y^2}{4}$

$\Rightarrow y \in \mathbb{R} \quad \& \quad z \leq 0$

$\Rightarrow$ Domain$(f) = \{(x,y) : x, y \in \mathbb{R}\} = \mathbb{R}^2$

Rng$(f) = \{z : z \in \mathbb{R}\} = (-\infty, \infty)$

Let's graph in Mma now

Next, we notice that there is only one intersect with all three coordinate axis at point $(0,0,0)$.

Finally, we can find our favorite traces

$z = z_0$ - trace: $x^2 - \frac{y^2}{4} = z_0$ Try $z_0 = 4 = 2^2$

$$\Rightarrow \quad x^2 - \frac{y^2}{2^2} = 2^2$$

$$\Rightarrow \quad \frac{x^2}{2^2} = \frac{y^2}{4^2} + 1$$

$$\Rightarrow \quad \frac{y^2}{4^2} = \frac{x^2}{2^2} - 1$$

$$\Rightarrow \quad y^2 = 4x^2 - 4^2 = 4x^2 - 16$$

Hyperbola's "parallel" to x-axis

xy-trace: $x^2 - \frac{y^2}{4} = 0 \Rightarrow y^2 = 4x^2$
(set $z=0$)

$$\Rightarrow \quad \sqrt{y^2} = \sqrt{4x^2}$$

$$\Rightarrow \quad |y| = 2|x|$$

$$\Rightarrow \quad y = \pm 2x$$