1. T (F) Let $c \in \mathbb{R}$ and let $\mathbf{x} \in \mathbb{R}^2$. Then $\|c\mathbf{x}\|_2 = c\,\|\mathbf{x}\|_2$.

Note: The true statement should be in the form

$$\|c \cdot \vec{x}\|_2 = \underbrace{|c|}_{\text{(needs an absolute value on c)}} \cdot \|\vec{x}\|_2$$

Counter Example: Let $c = -1$ and $\vec{x} = \langle 3, 4 \rangle$

$$\Rightarrow c \cdot \vec{x} = -1 \cdot \langle 3, 4 \rangle$$

$$= \langle -1 \cdot 3, -1 \cdot 4 \rangle$$

$$= \langle -3, -4 \rangle$$

$$\Rightarrow \|c \cdot \vec{x}\|_2 = \sqrt{(-3)^2 + (-4)^2}$$

$$= \sqrt{9 + 16}$$

$$= \sqrt{25}$$

$$\Rightarrow \|c \cdot \vec{x}\|_2 = 5$$

On the other hand, we have $\vec{x} = \langle 3, 4 \rangle$ and

$$\|\vec{x}\|_2 = \sqrt{3^2 + 4^2}$$

$$= \sqrt{9 + 16} = \sqrt{25} = 5$$

Problem 1, Counter Example Continued ....)

$$\Rightarrow \quad c \cdot \|\vec{x}\|_2 = -1 \cdot 5 = -5$$

Thus we see $5 = \|c \cdot \vec{x}\|_2 \neq c \cdot \|\vec{x}\|_2 = -5$

in this case. Since we have found a scalar $c = -1$

and vector $\vec{x} = \langle 3, 4 \rangle \in \mathbb{R}$ such that

$$\|c \cdot \vec{x}\|_2 \neq c \cdot \|\vec{x}\|_2$$

the statement in the problem must be false.

Problem 1...

General Proposition: Let $c \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^2$

$$\Rightarrow \quad \vec{x} = \langle x_1 , y_1 \rangle \quad \text{for} \quad x_1, y_1 \in \mathbb{R}$$

$$\Rightarrow \quad c \cdot \vec{x} = c \cdot \langle x_1 , y_1 \rangle$$

$$= \langle c \cdot x_1 , c \cdot y_1 \rangle$$

$$\Rightarrow \|c \cdot \vec{x}\|_2 = \sqrt{(c \cdot x_1)^2 + (c \cdot y_1)^2}$$

$$= \sqrt{c^2 \cdot x_1^2 + c^2 \cdot y_1^2}$$

$$= \sqrt{c^2 \cdot (x_1^2 + y_1^2)}$$

$$= \sqrt{c^2} \cdot \underbrace{\sqrt{x_1^2 + y_1^2}}_{\text{this is the 2-norm of the vector } \vec{x}}$$

$$= \sqrt{c^2} \cdot \|\vec{x}\|_2$$

Recall: $\sqrt{c^2} = |c|$

$$= |c| \cdot \|\vec{x}\|_2$$

WARNING: $\sqrt{c^2} \neq c$

Try $c = -1$: $\sqrt{(-1)^2} = \sqrt{1} = 1 \neq c$

2. Find the vector in the direction of $\langle 10, 24 \rangle$ with length 4.

Solution: Let $\vec{x} = \langle 10, 24 \rangle$. Then, we see

$$\|\vec{x}\|_2 = \sqrt{10^2 + 24^2}$$

$$= \sqrt{100 + 576}$$

$$= \sqrt{676}$$

$$= \sqrt{26^2}$$

$$= 26$$

Then, we can create a unit vector (a vector of length 1) by normalizing $\vec{x}$ (dividing $\vec{x}$ by it's norm). Let

$$\vec{u} = \frac{\vec{x}}{\|\vec{x}\|_2} = \frac{\langle 10, 24 \rangle}{26}$$

Note: $\dfrac{\vec{x}}{\|\vec{x}\|_2} = \underbrace{\dfrac{1}{\|\vec{x}\|_2}}_{\text{scalar}} \cdot \underbrace{\vec{x}}_{\text{vector}}$

scalar-vector multiplication

$$= \frac{1}{26} \cdot \langle 10, 24 \rangle$$

$$= \left\langle \frac{10}{26}, \frac{24}{26} \right\rangle = \left\langle \frac{5}{13}, \frac{12}{13} \right\rangle$$

Problem 2 ...)

Check $\|\vec{u}\|_2 = \sqrt{\left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2}$

$$= \sqrt{\frac{25}{169} + \frac{144}{169}}$$

$$= \sqrt{\frac{(25+144)}{169}}$$

$$= \sqrt{\frac{169}{169}}$$

$$= \sqrt{1}$$

$$= 1$$

← indeed: $\|\vec{u}\|_2 = 1$ and we've confirmed $\vec{u}$ is a unit vector.

Recall from problem 1 on this handout, if $c \in \mathbb{R}$ and $\vec{u} \in \mathbb{R}^2$, then $\|c \cdot \vec{u}\|_2 = |c| \cdot \|\vec{u}\|_2$. Thus, let's choose $c = 4$ and create a new vector

$$\vec{v} = c \cdot \vec{u} = 4 \cdot \vec{u}$$

We can confirm $\|\vec{v}\|_2 = 4$ in two ways.

Method 1: $\|\vec{v}\| = \|4 \cdot \vec{u}\|_2$

$$= |4| \cdot \underbrace{\|\vec{u}\|_2}$$

recall: $\vec{u}$ is a unit vector with $\|\vec{u}\|_2 = 1$

$$= 4 \cdot 1$$

$$= 4.$$

Method 2: $\vec{v} = 4 \cdot \vec{u} = 4 \cdot \langle \frac{5}{13}, \frac{12}{13} \rangle$

$$= \langle 4 \cdot \frac{5}{13}, 4 \cdot \frac{12}{13} \rangle$$

$$= \langle \frac{20}{13}, \frac{48}{13} \rangle.$$

$$\Rightarrow \|\vec{v}\|_2 = \sqrt{\left(\frac{20}{13}\right)^2 + \left(\frac{48}{13}\right)^2}$$

$$= \sqrt{\frac{400}{169} + \frac{2304}{169}}$$

$$= \sqrt{\frac{2704}{169}} = \frac{\sqrt{2704}}{\sqrt{169}} = \frac{52}{13} = \boxed{4.}$$

3. Consider the following three vectors

$$\mathbf{u} = \langle 4, -9 \rangle, \qquad \mathbf{v} = \langle -5, 9 \rangle, \qquad \mathbf{w} = \langle -2, -7 \rangle.$$

Which vector has the greater magnitude: $\mathbf{u} - \mathbf{v}$ or $\mathbf{w} - \mathbf{u}$.

Solution: Consider $\vec{u} - \vec{v} = \langle 4, -9 \rangle - \langle -5, 9 \rangle$

$$= \langle 4, -9 \rangle + -1 \cdot \langle -5, 9 \rangle$$

$$= \langle 4, -9 \rangle + \langle 5, -9 \rangle$$

$$= \langle 4 + 5, -9 + -9 \rangle$$

$$= \langle 9, -18 \rangle$$

$$\Rightarrow \quad \| \vec{u} - \vec{v} \|_2 = \sqrt{9^2 + (-18)^2}$$

$$= \sqrt{81 + 324}$$

$$= \sqrt{405}$$

Note: $405 = 81 \cdot 5$

$$= \sqrt{81 \cdot 5}$$

$$= \sqrt{81} \cdot \sqrt{5} = 9 \cdot \sqrt{5} \approx 20.1246$$

Problem 3, continued ...

On the other hand, consider the vector difference

$$\vec{w} - \vec{u} = \langle -2, -7 \rangle - \langle 4, -9 \rangle$$

$$= \langle -2 - 4, -7 + 9 \rangle$$

$$= \langle -6, 2 \rangle$$

$$\Rightarrow \|\vec{w} - \vec{u}\|_2 = \sqrt{(-6)^2 + (2)^2}$$

$$= \sqrt{36 + 4}$$

$$= \sqrt{40}$$

$$= \sqrt{4 \cdot 10}$$

$$= \sqrt{4} \cdot \sqrt{10}$$

$$= 2 \cdot \sqrt{10} \approx 6.32456$$

Thus, we see that $\|\vec{w} - \vec{u}\|_2 < \|\vec{u} - \vec{v}\|_2$ and $\vec{u} - \vec{v}$ has greater magnitude

4. Let $\mathbf{u} = \langle a, 5\rangle$ and $\mathbf{v} = \langle 3, 7\rangle$

a. Find the value of parameter a such that $\mathbf{u}$ is parallel to $\mathbf{v}$

b. As we will see in lesson 3, two nonzero vectors $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ are orthogonal if and only if $u_1v_1 + u_2v_2 = 0$. Find the value of a such that $\mathbf{u}$ is orthogonal to $\mathbf{v}$.

Solution to Problem 4, part a)

Recall that for two vectors in $\mathbb{R}^2$, we say $\vec{u}$ is parallel to $\vec{v}$ if and only if (iff) $\vec{u}$ is in the "same direction" as $\vec{v}$. Moreover, we say $\vec{u}$ is in the same direction of $\vec{v}$ iff we can write

$$\vec{u} = c \cdot \vec{v} \quad \text{for some } c \in \mathbb{R}$$

$$\Rightarrow \quad \langle a, 5\rangle = c \cdot \langle 3, 7\rangle$$

$$\Rightarrow \quad \langle a, 5\rangle = \langle 3c, 7c\rangle$$

$$\Rightarrow \quad \underbrace{3c = a}_{\text{equation 1 from component 1}} \quad \text{and} \quad \underbrace{5 = 7 \cdot c}_{\text{equation 2 from component 2}}$$

Solution to Problem 4, part a continued...

$$\Rightarrow \quad c = \frac{5}{7} \qquad \text{by equation 2}$$

$$\Rightarrow \quad 3c = \frac{3 \cdot 5}{7} = \boxed{\frac{15}{7} = a}$$

$\Rightarrow$ If $\vec{u} = \langle \frac{15}{7}, 5 \rangle$ and $\vec{v} = \langle 3, 7 \rangle$

then $\vec{u} = c \cdot \vec{v}$ for $c = \frac{5}{7}$ and we see

$\vec{u}$ and $\vec{v}$ are in the same direction (they

are parallel).

Solution to problem 4, part b).

By the problem statement, we know vectors

$$\vec{u} = \langle u_1, u_2 \rangle \quad \text{and} \quad \vec{v} = \langle v_1, v_2 \rangle$$

are orthogonal iff $\quad 0 = u_1 \cdot v_1 + u_2 \cdot v_2$.

In this case, we have

$$\vec{u} = \langle a, 5 \rangle \quad \text{and} \quad \vec{v} = \langle 3, 7 \rangle$$

$$\Rightarrow \quad \vec{u} \perp \vec{v} \quad \Leftrightarrow \quad 3 \cdot a + 5 \cdot 7 = 0$$

($\vec{u}$ is orthogonal to $\vec{v}$)

$$\Leftrightarrow \quad 3a + 35 = 0$$

$$\Leftrightarrow \quad 3a = -35$$

$$\Leftrightarrow \quad a = \frac{-35}{3}$$

$$\Leftrightarrow \quad \vec{u} = \langle -\tfrac{35}{3}, 5 \rangle \text{ is orthogonal}$$

$$\text{to } \vec{v} = \langle 3, 7 \rangle$$