Math 1C, Part II: Sequences and Series

In part I of Math 1C, we studied multivariable differentiation. We saw that these techniques generalized the work we did in single-variable calculus to functions with multiple variables:

	Single Variable Calculus	Multiple Variable Calculus
Differentiation (Forward Problem)	Math 1A $\frac{d}{dx}\Big[\underbrace{F(x)}_{\text{given single variable function}}\Big] = \underbrace{f(x)}_{\text{unknown ordinary derivative}}$ ordinary derivative operator	Math 1C (Part I) $\nabla\Big[\underbrace{F(x,y)}_{\text{given multi-variable function}}\Big] = \Big\langle \underbrace{F_x(x,y),\ F_y(x,y)}_{\text{unknown partial derivatives}} \Big\rangle$ gradient operator
Integration (Backward Problem)	Math 1B $\frac{d}{dx}\Big[\underbrace{F(x)}_{\text{unknown antiderivative}}\Big] = \underbrace{f(x)}_{\text{known derivative function (integrand)}}$	Math 1D $\nabla\Big[\underbrace{F(x,y)}_{\text{unknown potential function}}\Big] = \Big\langle \underbrace{F_x(x,y),\ F_y(x,y)}_{\text{known vector field}} \Big\rangle$

In part II of Math 1C, we will go back to our discussion of single variable functions. Specifically, given a single-variable function $f(x)$, we will develop theory (Known as Taylor series polynomials) that enables us to represent $f(x)$ as an infinite sum of powers of $(x-a)$:

this equality will hold for specific x-values

$$\underbrace{f(x)}_{\text{explicit representation of function } f(x)} \overset{\downarrow}{=} \underbrace{\sum_{n=0}^{\infty} c_n \cdot (x-a)^n}_{\text{equivalent power series representation for function}}$$

Recall: Each term in this sum is given as

$$\underset{\text{nth coefficient}}{c_n} \cdot \underbrace{(x-a)^n}_{\text{nth power of } (x-a)}$$

The idea of representing a function in multiple ways has analogs to your studies in previous classes.

Multiple Representations of Real Numbers

$$36 = 6^2$$

36 — explicit representation of number

6^2 — representation of number as a perfect square

$$= 2^2 \cdot 3^2$$

representation of number using it's prime factorization

$$= \sqrt[2]{1296}$$

representation of number as the sqrt of it's square

$$= \frac{72}{2}$$

representation of number as an equivalent, unsimplified fraction

$$= 36 + 0$$

All of these are valid representations of the number 36. Each of the representations is used for a specific purpose that arises when solving problems involving the number 36. The specific form we choose to work with will depend on our goal.

Our study of functions up through Math 1A/1B was centrally focused on developing deep conceptual understanding of function notation, the input / output relationships in functions, developing a library of well-known functions (known as the elementary functions which include polynomials, trig functions, rational functions, radical functions, exponential functions, logaritmic functions, abs values and any function formed using operations between these functions), and using this knowledge to construct a formal definition of ordinary derivatives.

Because one of the major goals of your early math education was to maximize your understanding of and familiarity with function notation, you have thus far studied very specific representations of the functions you know and love. For example, thus far you have seen two representations of the sine function:

$$f(x) = \underbrace{\sin(x)}_{\text{explicit representation}} = \underbrace{\frac{\text{opposite}}{\text{hypotenuse}}}_{\text{geometric interpretation}}$$

In Math 1C (and beyond), you now have a sophisticated grasp of the basic concepts behind function notation. Thus, we will begin to develop new representations of the functions you know and love to solve more challenging problems. These problems include:

I. The Numerical Evaluation Problem

Given an elementary function $f(x)$ and a constant $c \in \text{Dom}(f)$, calculate a numerical approximation for the output value of $f(c)$, call it $\hat{f}_c$, that is as accurate as you want:

$$| f(c) - \hat{f}_c | < \varepsilon$$

exact value of $f(x)$ at $x = c$

(often this is impossible to find exactly)

numerical approximation to $f(c)$

desired approximation accuracy

(on your TI 84, this is approximately $\varepsilon \approx 10^{-14}$

(the display is accurate up to 10 digits)

More about this available in Math 2A

II. Ordinary Differential Equations Problem

Given an ordinary differential equation that models a phenomenon that you want to study

find a "solution" to this equation.

When attempting to solve these problems, a very useful set of results come from representing a function $f(x)$ as a power series, as we shall see.

Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 8.1 p. 596 - 606 Section 8.2 p. 607 - 619

Definition. *p. 597* ***Sequence***

A **sequence** $\{a_n\}_{n=1}^{\infty}$ is a function $a : \mathbb{N} \longrightarrow \mathbb{R}$. In other words, a sequences is an ordered list of numbers. We can write any sequence as

i. An **ordered set** as $\{a_1, a_2, a_3, a_4, ..., a_n, ...\}$. The term a_1 is called the *first term*, a_2 is called the *second term*, and in general a_n is called the *nth term.*

ii. A **recurrence relation** of the form $a_{n+1} = f(a_n)$ for $n \in \mathbb{N}$ where a_1 must be given.

iii. An **explicit formula** of the form $a_n = f(n)$ for $n \in \mathbb{N}$.

Example 8.1.2 p. 598) Suppose we have a sequence $\{a_n\}_{n=1}^{\infty}$ defined by the recurrence relation

$$a_1 = 1 \qquad a_{n+1} = 2 \cdot a_n + 1$$

Find the first 6 terms of this sequence.

Then find an explicit formula for a_n.

Solution: Let's consider the table of values

n	a_n (assuming $a_1 = 1$)	$a_n - a_{n-1}$
1	$\boxed{a_1 = 1}$	
2	$a_2 = 2 \cdot a_1 + 1$ $= 2 \cdot 1 + 1$ $\Rightarrow \boxed{a_2 = 3}$	$3 - 1 = 2$
3	$a_3 = 2 \cdot a_2 + 1$ $= 2 \cdot 3 + 1$ $\Rightarrow \boxed{a_3 = 7}$	$7 - 3 = 4$

n	a_n (assuming $a_1 = 1$)	$a_n - a_{n-1}$
4	$a_4 = 2 \cdot a_3 + 1 = 2 \cdot 7 + 1$ $\Rightarrow \boxed{a_4 = 15}$	$15 - 7 = 8$
5	$a_5 = 2 \cdot a_4 + 1 = 2 \cdot 15 + 1$ $\Rightarrow \boxed{a_5 = 31}$	$31 - 15 = 16$
6	$a_6 = 2 \cdot a_5 + 1 = 2 \cdot 31 + 1$ $\Rightarrow \boxed{a_6 = 63}$	$63 - 31 = 32$

Now, let's study the pattern(s) we see in this sequence

n	a_n
1	1
2	3
3	7
4	15
5	31
6	63

Upon further inspection, this sequence looks a lot like the powers of 2

n	2^n
1	2
2	4
3	8
4	16
5	32
6	64

Then, we can attempt to find an explicit formula for a_n.

We guess

$$a_n = 2^n - 1$$

Let's test our guess:

Explicit: $a_7 = 2^7 - 1 = 128 - 1 = 127$ ✓

Recursive: $a_7 = 2 \cdot a_6 + 1 = 2 \cdot 63 + 1 = 126 + 1 = 127$ ✓

Then, we might try to find a proof that if $a_1 = 1$ and $a_{n+1} = 2 \cdot a_n + 1$. This is called a proof by induction and is included on the next page for completeness.

Base case $n = 1$: $a_1 = 2^1 - 1 = 1$ ✓

Induction hypothesis: Assume that for all $n \in \mathbb{N}$,

$$a_n = 2^n - 1$$

Induction step: Consider

$$a_{n+1} = 2 \cdot a_n + 1$$

$$= 2 \cdot (2^n - 1) + 1$$

$$= 2^1 \cdot 2^n - 2 + 1$$

$$= 2^{n+1} - 1$$

Thus we see by induction that $a_n = 2^n - 1$ ✓

Example 8.1.3a) Let $\{-2, 5, 12, 19, \dots\}$ represent an infinite sequence. Let's

A. Find the next two terms of this sequence

B. Write the recurrence relation that generates this sequence

C. Find an explicit formula for the nth term of the sequence.

Solution: Let's begin by mapping n and a_n

n	a_n	$a_n - a_{n-1}$
1	$a_1 = -2$	
2	$a_2 = 5$	$+7$
3	$a_3 = 12$	$+7$
4	$a_4 = 19$	$+7$

5	$a_5 = 26$
6	$a_6 = 33$

Based on our work, we see

$$a_2 = a_1 + 7 = -2+7 = 5$$

$$a_3 = a_2 + 7 = 5 + 7 = 12$$

$$a_4 = a_3 + 7 = 12 + 7 = 19$$

We guess that our general recurrence relation takes the form

$$\boxed{a_1 = -2, \quad a_{n+1} = a_n + 7.}$$

To get an explicit formula for a_n, we note

$$a_1 = -2$$

$$a_2 = -2 + 7 = 5$$

$$a_3 = 5 + 7 = -2 + 7 + 7 = -2 + \underbrace{2 \cdot 7} = 12$$

$$a_4 = 12 + 7 = -2 + 2 \cdot 7 + 7 = -2 + \underbrace{3 \cdot 7}$$

$$\Rightarrow \boxed{a_n = -2 + 7 \cdot (n-1)}$$

Example 8.1.3b p. 598) Let $\{b_n\}_{n=1}^{\infty} = \{3, 6, 12, 24, 48, \ldots\}$ be an infinite sequence. Let's

A. Find the next two terms of our sequence.

B. Write a recurrence relation for b_n

C. Find an explicit formula for b_n

Solution: Let's begin by mapping n to b_n

n	b_n	$b_n - b_{n-1}$
1	$b_1 = 3$	
2	$b_2 = 6$	$6 - 3 = 3$
3	$b_3 = 12$	$12 - 6 = 6$
4	$b_4 = 24$	$24 - 12 = 12$
5	$b_5 = 48$	$48 - 24 = 24$
		$\vdots$
6	$b_6 = 96$	48
7	$b_7 = 192$	96

Based on this work we see

$$b_2 = 2 \cdot b_1 = 2 \cdot 3 = 6$$

$$b_3 = 2 \cdot b_2 = 2 \cdot 6 = 12$$

$$b_4 = 2 \cdot b_3 = 2 \cdot 12 = 24$$

$$b_5 = 2 \cdot b_4 = 2 \cdot 24 = 48$$

$$\Rightarrow \boxed{b_1 = 3 \quad , \quad b_{n+1} = 2 \cdot b_n}$$

To obtain a general formula, we notice

$$a_1 = 3 = 3 \cdot 1 = 3 \cdot 2^0$$

$$a_2 = 2 \cdot 3 = 2^1 \cdot 3$$

$$a_3 = 2 \cdot 6 = 2 \cdot (2 \cdot 3) = 2^2 \cdot 3$$

$$a_4 = 2 \cdot 12 = 2 \cdot (2^2 \cdot 3) = 2^3 \cdot 3$$

$$a_5 = 2 \cdot 24 = 2 \cdot (2^3 \cdot 3) = 2^4 \cdot 3$$

$$\Rightarrow \boxed{a_n = 3 \cdot 2^{n-1}} \quad \text{for } n = 1, 2, 3, 4$$

Definition. *p. 599* ***Limit of a Sequence***

If the terms of sequence $\{a_n\}_{n=1}^{\infty}$ approach a unique number L as n increases, we say

$$\lim_{n\to\infty} a_n = L.$$

That is to say, if a_n can be made arbitrarily close to L by taking n "sufficiently" large, then we say the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ is L.

If $\lim\limits_{n\to\infty} a_n = L$, then we say that $\{a_n\}_{n=1}^{\infty}$ **converges** to L.

If the terms of the sequence $\{a_n\}_{n=1}^{\infty}$ do not approach a single number as n increases, we say the sequences has no limit and the sequence **diverges**.

Theorem 8.1. *p. 607* ***Limits of Sequences from Limits of Functions***

Suppose $f(x)$ is a function such that $f(n) = a_n$ for all $n \in \mathbb{N}$. If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ is given by $\lim_{n\to\infty} a_n = L$.

Example 8.2.1 a p. 608) Let $a_n = \dfrac{3n^3}{n^3+1}$. Find $\lim_{n\to\infty} a_n$.

Solution: If $a_n = \dfrac{3n^3}{n^3+1}$, then we can define

the function $f(x) = \dfrac{3x^3}{x^3+1}$. Note: $f(n) = a_n$.

Moreover, we know by Math 1A that

$$\lim_{x\to\infty} \frac{3x^3}{x^3+1} = \lim_{x\to\infty} \frac{3}{1 + 1/x^3} = 3$$

Then, by thm 8.1, we see $\boxed{\lim_{n\to\infty} a_n = 3}$. ✓

Theorem 8.2. *p. 607* ***Limit Laws for Sequences***

Suppose that $c \in \mathbb{R}$ and p is a positive integer. Suppose that the limits

$$\lim_{n\to\infty} a_n = A \qquad \text{and} \qquad \lim_{n\to\infty} b_n = B$$

exist. Then, as long as we check these conditions, we can conclude

1. *Sum Law:* $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = A + B$

2. *Difference Law:* $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = A - B$

3. *Constant Multiple Law:* $\lim_{n\to\infty} (c \cdot a_n) = c \cdot \lim_{n\to\infty} a_n = c \cdot A$

4. *Product Law:* $\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) = A \cdot B$

5. *Quotient Law:* $\lim_{n\to\infty} \left[\frac{a_n}{b_n}\right] = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$ if $\lim_{n\to\infty} b_n = B \neq 0$

6. *Constant Law:* $\lim_{n\to\infty} c = c$

7. *General Root Law:* $\lim_{n\to\infty} \left[\ a_n^{\,p}\ \right] = \left[\ \lim_{n\to\infty} a_n\ \right]^p = A^p$ (if $p > 0$ and $a_n > 0$.)

Example 8.1.5 p. 599) Let's define the sequence

$$a_n = \frac{4 \cdot n^3}{n^3 + 1}$$

Find $\lim_{n \to \infty} a_n$.

Solution: To find our desired limit, we might consider a table of values. We can also do some algebraic manipulations

$$a_n = \frac{4 \cdot n^3}{n^3 + 1}$$

$$= \frac{4 \cdot n^3}{(n^3 + 1)} \cdot \frac{1/n^3}{1/n^3}$$

$$= \frac{4}{1 + 1/n^3}$$

$$\Rightarrow \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{4}{1 + 1/n^3} = 4.$$

Definition. *p. 608* ***Terminology for Sequences***

$\{a_n\}_{n=1}^{\infty}$ is **increasing** if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$

$\{a_n\}_{n=1}^{\infty}$ is **nondecreasing** if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$

$\{a_n\}_{n=1}^{\infty}$ is **decreasing** if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$

$\{a_n\}_{n=1}^{\infty}$ is **nonincreasing** if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$

$\{a_n\}_{n=1}^{\infty}$ is **monotonic** if it is either nonincreasing or nondecreasing

$\{a_n\}_{n=1}^{\infty}$ is **bounded** if there is a real number $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$

Example p. 609) Let $a_n = 1 - \frac{1}{n} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\}$.

Notice $a_n = 1 - \frac{1}{n} = \frac{n-1}{n}$.

Since $n-1 \leq n$ for all $n \in \mathbb{N}$, we have

$$\frac{n-1}{n} \leq 1 \Rightarrow |a_n| \leq 1$$

$$\Rightarrow a_n \text{ bounded.}$$

Moreover, we see $a_{n+1} \geq a_n$ and thus a_n is increasing since

$$a_{n+1} = \frac{n}{n+1} \quad \text{and} \quad a_n = \frac{n-1}{n}$$

We know that $n \cdot n \geq (n-1)\cdot(n+1) = n^2 - 1$

$$\Rightarrow \frac{n}{n+1} \geq \frac{n-1}{n}$$

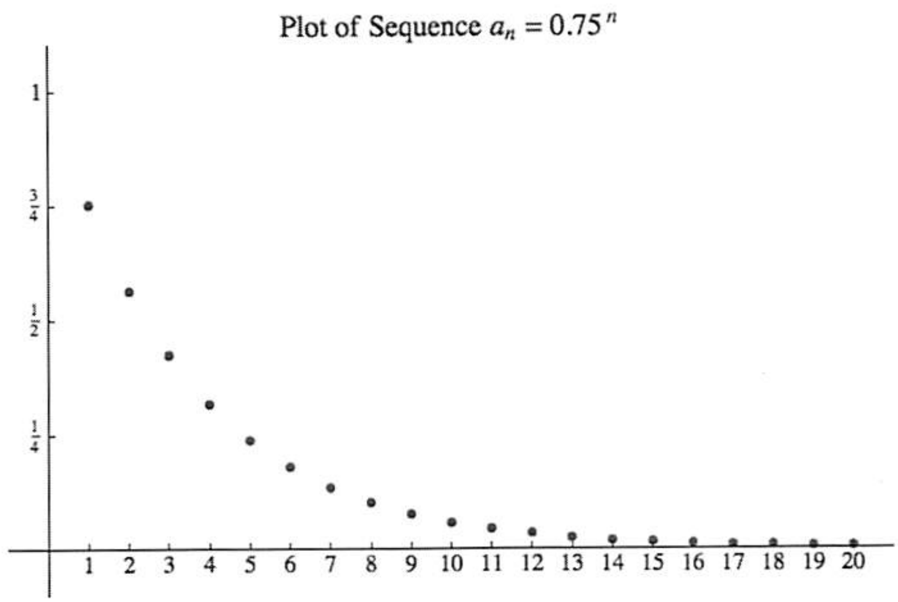
$$\Rightarrow a_{n+1} \geq a_n \checkmark$$

Definition. p. 609 ***Geometric Sequence***

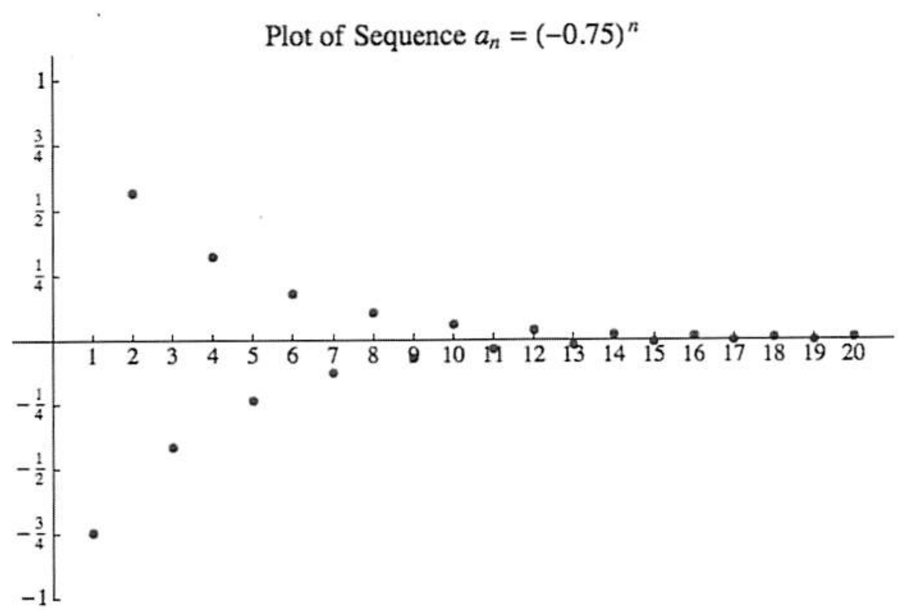
We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is a **geometric sequence** if it has the property that each term a_{n+1} is obtained by multiplying the previous term a_n by a fixed constant r. In other words, geometric sequences have the property that $a_{n+1} = ra_n$. The term r is call the **ratio** of the sequence. Further, all geometric sequences can be written with general formula $a_n = a \cdot r^{n-1}$ for some constant $a \in \mathbb{R}$.

Example 8.2.3 p. 610) Let's consider $a_n = r^n$ for different values of ratio r.

$r = \frac{3}{4}$

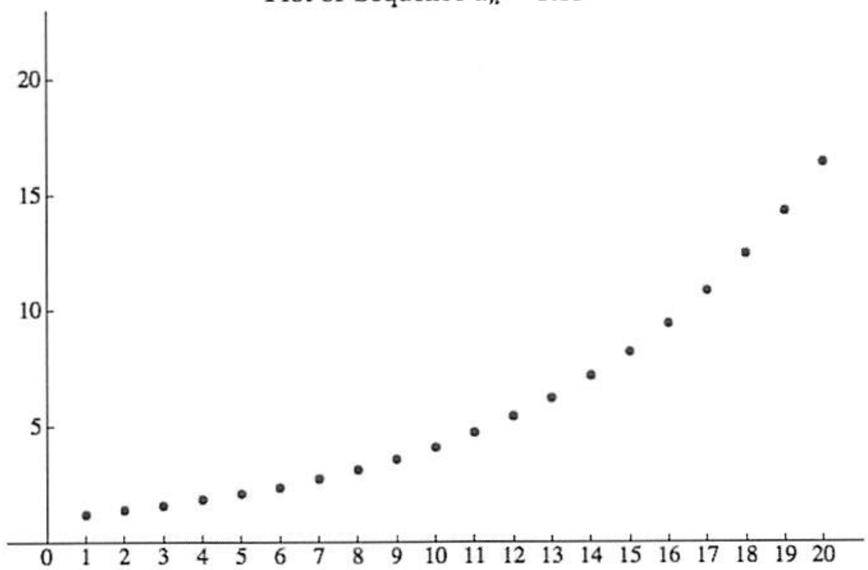


$r = -\frac{3}{4}$



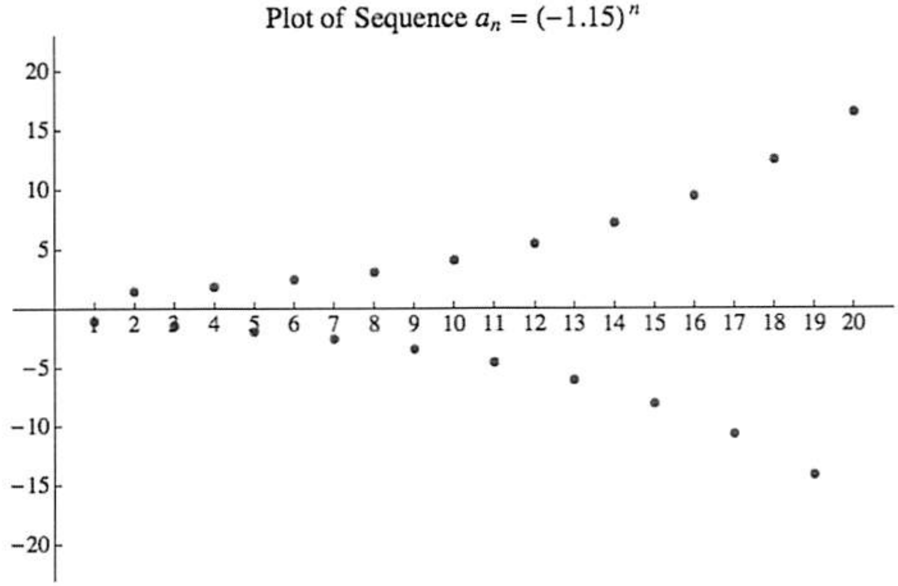
Plot of Sequence $a_n = 1.15^n$

r = 1.15



Plot of Sequence $a_n = (-1.15)^n$

r = -1.15



What patterns do we notice?

- All of these sequences take the same form:

$$a_n = r^n \qquad \text{for some} \quad r \in \mathbb{R}$$

- In the case that $0 \leq r < 1$, we see

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} r^n = 0$$

- If we have $-1 < r \leq 0$, we can write

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} r^n$$

$$= \lim_{n\to\infty} (-1 \cdot \rho)^n \qquad \text{for some} \quad 0 \leq \rho < 1$$

this is the greek letter "rho"

$$= \lim_{n\to\infty} (-1)^n \cdot \rho^n$$

We note that $-\rho^n \leq (-1)^n \cdot \rho^n \leq \rho^n$ for all $n \in \mathbb{N}$

and that $(-1)^n$ alternates between -1 and 1. Thus this limit goes to zero.

□ In the case that $r > 1$, we see that

$$1 < r < r^2 < r^3 < \cdots < r^n$$

and the sequence $a_n = r^n$ is strictly increasing without an upper bound. Thus, this sequence diverges.

□ In the case that $r < -1$, we can write

$$r^n = (-1 \cdot \rho)^n \qquad \text{for } 1 < \rho$$

$$= (-1)^n \cdot \rho^n$$

We know $(-1)^n$ alternates between -1 and $+1$ for $n \in \mathbb{N}$ and ρ^n increases without bound.

We can summarize these observations in a theorem.

Theorem 8.3. *p. 611* ***Limit of a geometric sequence***

Let $r \in \mathbb{R}$ be a real number. Then

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \le -1 \text{ or } r > 1 \end{cases}$$

If $r > 0$, then the sequences $\{r^n\}_{n=1}^{\infty}$ is a monotonic sequence. if $r < 0$, then the sequence $\{r^n\}_{n=1}^{\infty}$ oscillates.

Proof: Let's prove this theorem using the three cases provided for us:

Case 1 : $r = 1$

Case 2 : $0 \le r < 1$ or $-1 < r < 0$

Case 3: $r > 1$ or $r \le -1$

Case 1: Let $r = 1 \Rightarrow r^n = 1$ for all $n \in \mathbb{N}$

$$\Rightarrow \lim_{n\to\infty} r^n = \lim_{n\to\infty} 1 = 1. \quad \square$$

Case 2 : Let $0 \leq r < 1$

□ If $r = 0$, then $r^n = 0^n = 0$ and

we see $\lim_{n\to\infty} r^n = \lim_{n\to\infty} 0 = 0$ ✓

□ If $0 < r < 1$, then let $M = \frac{1}{r} - 1 > 0$

For any $\varepsilon > 0$, we know there is a $N \in \mathbb{N}$

such that $\frac{1}{\varepsilon \cdot M} < N$.

Moreover, we have

$$r^n = \frac{1}{(1+M)^n}$$

$M = \frac{1}{r} - 1 \Rightarrow \frac{1}{r} = M + 1$

$\Rightarrow r = \frac{1}{M+1}$

$$\leq \frac{1}{1 + n \cdot M}$$

By the Binomial thm, we have

$(1+M)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot M^k \geq 1 + n \cdot M$

$$\leq \frac{1}{n \cdot M}$$

$\Rightarrow \frac{1}{(1+M)^n} \leq \frac{1}{1 + n \cdot M}$

$$\leq \frac{1}{N \cdot M} < \varepsilon .$$

□ If $-1 < r < 0$, then $r = -1 \cdot \rho$ for $0 < \rho < 1$.

Moreover, we can write

$$-\rho^n \leq r^n \leq \rho^n$$

By our previous reasoning, we know

$$\lim_{n \to \infty} -\rho^n = \lim_{n \to \infty} \rho^n = 0$$

Thus, by the squeeze theorem, we have $\lim_{n \to \infty} r^n = 0$.

Case 3: Left as an exercise.

Theorem 8.4. *p. 611* ***The Squeeze Theorem***

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ be sequences with $a_n \leq b_n \leq c_n$ for all integers n greater than some index $N \in \mathbb{N}$. If

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim_{n\to\infty} b_n = L$.

Example 8.2.4 p. 611) Find the limit of the sequence

$$b_n = \frac{\cos(n)}{n^2+1}$$

Solution: We know that

$$-1 \leq \cos(n) \leq 1$$

$$\Rightarrow \quad \frac{-1}{n^2+1} \leq \frac{\cos(n)}{n^2+1} \leq \frac{1}{n^2+1}$$

$\Rightarrow$ if $a_n = \dfrac{-1}{n^2+1}$ and $c_n = \dfrac{1}{n^2+1}$

then $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} c_n$,

we see $\lim_{n\to\infty} b_n = 0$ by squeeze theorem.

Theorem 8.5. *p. 612* ***Bounded Monotonic Sequences***

Every bounded, monotonic sequence is convergent.

Exercise 8.2.45 p. 616) Let $a_n = 0.2^n = \left[\frac{1}{5}\right]^n$. Then, for all

$n \in \mathbb{N}$, we note that

- $|a_n| \leq 1 = M$ — this sequence is bounded

- Since $0 < \frac{1}{5} < 1$ and $0 < \left[\frac{1}{5}\right]^n$ for any $n \in \mathbb{N}$,

  we know $\frac{1}{5} \cdot \left[\frac{1}{5}\right]^n < 1 \cdot \left[\frac{1}{5}\right]^n$

  $$\Rightarrow \left[\frac{1}{5}\right]^{n+1} < \left[\frac{1}{5}\right]^n$$

  $$\Rightarrow a_{n+1} < a_n$$

  $\Rightarrow$ a_n is decreasing

  $\Rightarrow$ a_n is monotanic

Since $\{a_n\}$ is bounded and monotonic, we know by thm 8.5 that $\{a_n\}_{n=1}^{\infty}$ is convergent.

$\boxed{\text{L15, p.29}}$