Math 1C: Calculus III

Lesson 13: Maximum and Minimum Problems

Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 12.8: Limits and Continuity, p. 939 - 951

> **Definition.** ***Local Maximum Value(s)*** *p. 939*
>
> Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. We say that f has a local maximum at (a, b) if and only if
>
> $$f(x, y) \leq f(a, b)$$
>
> for (x, y) in the domain of f in some open disk centered at (a, b). We call this output value $f(a, b)$ the **local maximum value** on this open disk $D \subseteq \mathbb{R}^2$ since
>
> $$f(a, b) = \max_{\mathbf{x} \in D} f(\mathbf{x})$$
>
> In this case, we call the input point (a, b) a **local maximizer** of the function f since
>
> $$(a, b) = \arg\max_{\mathbf{x} \in D} f(\mathbf{x})$$

Note on notation:

- The "maximum value" refers to the output value of our function.

- When we refer to a disk $D \subseteq \mathbb{R}^2$, we mean an open set

$$D = \{(x,y) : (x-a)^2 + (y-b)^2 < \delta\}$$

for some positive radius δ

- The argument of $f(\vec{x}) = f(x,y)$ is the input (x,y).
  argument
  Thus, the arg max $f(\vec{x})$ is the input point (x,y) of f that produces a max.

Definition. ***Local Minimum Value(s)*** *p. 939*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. We say that f has a local minimum at (a, b) if and only if

$$f(x, y) \geq f(a, b)$$

for (x, y) in the domain of f in some open disk centered at (a, b). We call this output value $f(a, b)$ the **local minimum value** on this open disk $D \subseteq \mathbb{R}^2$ since

$$f(a, b) = \min_{\mathbf{x} \in D} f(\mathbf{x})$$

In this case, we call the input point (a, b) a **local minimizer** of the function f since

$$(a, b) = \arg\min_{\mathbf{x} \in D} f(\mathbf{x})$$

arg min: argument minimizer gives the input values that produce the minimum

Quick Check 12.8.1 p. 939)

Consider the paraboloid

$$z = x^2 + y^2 - 4x + 2y + 5$$

Using the method of completing the square, find the local minimum.

Solution: Consider the paraboloid

$$z(x, y) = x^2 - 4x + y^2 + 2y + 5$$

we can "complete the square" in both variables x and y.

$$z(x,y) = x^2 - 4x + 4 + y^2 + 2y + 1 + 5 - 4 - 1$$

$$= (x-2)^2 + (y+1)^2$$

we see that $z(x,y)$ is the sum of squares

and we know

$(x-2)^2 \geq 0$ for all $x \in \mathbb{R}$

$(y+1)^2 \geq 0$ for all $y \in \mathbb{R}$

Then $z(x,y) \geq 0$ for all $(x,y) \in \mathbb{R}^2$

Moreover, we see

$$z(2,-1) = (2-2)^2 + (-1+1)^2 = 0$$

$$\Rightarrow \underbrace{(2,-1)}_{\text{local minimizer}} = \underset{(x,y)\in\mathbb{R}^2}{\arg\min}\ z(x,y) \quad \text{AND}$$

local minimum value $\longrightarrow$ $0 = \min_{(x,y)\in\mathbb{R}^2} z(x,y)$.

Theorem 12.13. ***Necessary conditions for unconstrained optimization problems*** p. *939*

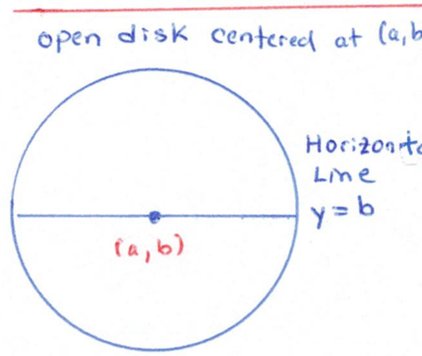
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. If f has a local maximum or local minimum at (a, b) and if $f(x, y)$ is differentiable at the point (a, b), then $\nabla f(a, b) = 0$ (i.e. $f_x(a, b) = f_y(a, b) = 0$).

Proof 1: Without loss of generality, suppose $f(x,y)$ has a local maximum value at (a,b).

Define the single variable function

$$g(x) = f(x, b)$$

obtained by holding $y = b$ constant.



By definition, there is some positive radius δ of an open disk

$$D = \{ (x,y) : \sqrt{(x-a)^2 + (y-b)^2} < \delta \}$$

such that $f(a,b) \geq f(x,y)$ for all $(x,y) \in D$

Then, we know $g(a) \geq f(x)$ for all $(x,b) \in D$

and $g(a)$ must be a local max of $g(x)$.

Since $f(x,y)$ is differentiable, we know $g(x)$ must also be differentiable and by Thm 4.2 on pg 239

we know

$$g'(a) = \frac{d}{dx}\left[g(x)\right]\Big|_{x=a} = 0$$

$$\Rightarrow \quad g'(a) = f_x(a,b) = 0$$

We repeat this argument with another single variable function defined by

$$h(y) = f(a,y)$$

obtained by holding $x = a$ constant. By the same reasoning, we see $h(b)$ is a local max and we know

$$0 = h'(b) = f_y(a,b).$$

Then, $\vec{\nabla} f(a,b) = \langle 0, 0 \rangle$ as was claimed.

Recall that for a function of one variable, the condition that $f'(a) = 0$ does not guarentee a local extreme value at input $x = a$.

Side comment

Consider $f(x) = x^3$.

We know $f'(x) = 3x^2$ and $f'(0) = 3 \cdot 0^2 = 0$. However $f(0)$ is neither a max nor min.

Theorem 12.13 is similar. The condition

$$\vec{\nabla} f(a,b) = \langle 0, 0 \rangle \iff \begin{array}{l} f_x(a,b) = 0 \quad \text{AND} \\ f_y(a,b) = 0 \end{array}$$

does not immediatly imply the point (a,b) producs a local extreme value of $f(x,y)$. Instead, Theorem 12.13 states that only points at which $\vec{\nabla} f = \vec{0}$ can be considered as "candidates" for local extrema.

Definition. ***Critical Point*** *p. 940*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. An interior point (a, b) in the domain of the function f is a **critical point** of f if and only if either of the following is true:

1. $\nabla f(a, b) = 0$ (i.e. the first partials $f_x(a, b) = f_y(a, b) = 0$)
2. at least one of the partial derivatives f_x or f_y does not exist at the point (a, b)

Example 12.8.1 p. 940) Find all critical points of

$$f(x,y) = x \cdot y \cdot (x-2) \cdot (y+3)$$

Solution: We begin by noticing

$$f_x(x,y) = \frac{\partial}{\partial x}\left[x \cdot (x-2) \cdot y(y+3)\right]$$

$$= y \cdot (y+3) \cdot \frac{\partial}{\partial x}\left[x^2 - 2x\right]$$

$$= y \cdot (y+3) \cdot (2x-2)$$

$$= 2y \cdot (x-1) \cdot (y+3)$$

This partial derivative is continuous at all $(x,y) \in \mathbb{R}^2$ since it is a polynomial in x & y.

Similarly, we have

$$f_y(x,y) = \frac{\partial}{\partial y}\left[x\cdot(x-2)\cdot y(y+3)\right]$$

$$= x\cdot(x-2)\ \frac{\partial}{\partial y}\left[y^2+3y\right]$$

$$= x\cdot(x-2)\cdot(2y+3)$$

which is also continuous for all $(x,y)\in\mathbb{R}^2$.

Thus, $\vec{\nabla} f(x,y)$ exists for all $(x,y)\in\mathbb{R}^2$ and the only type of critical points we have are of the flavor that $\vec{\nabla} f = \vec{0}$. Then, we need to find points (x,y) such that

$\boxed{\text{Eq I}}$ $$f_x(x,y) = 2y\cdot(x-1)(y+3) = 0$$ AND

$\boxed{\text{Eq II}}$ $$f_y(x,y) = x\cdot(x-2)\cdot(2y+3) = 0$$

By equation I, we know

$$f_x(x,y) = 2y \cdot (x-1) \cdot (y+3) = 0$$

$$\Rightarrow \quad 2y = 0 \quad \text{OR} \quad x-1 = 0 \quad \text{OR} \quad y+3 = 0$$

$$\Rightarrow \quad y = 0 \quad \text{OR} \quad x = 1 \quad \text{or} \quad y = -3$$

Let's consider each case separately:

Case $y=0$: Substituting condition $y=0$ into equation II, we have

$$f_y(x,y) = x \cdot (x-2) \cdot (2y+3) = 0$$

$$\Rightarrow \quad 3x \cdot (x-2) = 0$$

$$\Rightarrow \quad 3x = 0 \quad \text{or} \quad x-2 = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = 2$$

$\Rightarrow$ We have two critical points associated with this case, given by

$$\boxed{(0,0)} \quad \text{OR} \quad \boxed{(2,0)}$$

Case $x=1$: Substituting the condition $x=1$ into equation II, we have

$$f_y(x,y)\Big|_{x=1} = 1\cdot(1-2)(2y+3) = 0$$

$$\Rightarrow \quad -(2y+3)=0$$

$$\Rightarrow \quad y = -3/2$$

$\Rightarrow$ We have one critical point associated with this case, given by $\boxed{(1,-3/2)}$

Case $y=-3$: Substituting $y=-3$ into equation II, we have

$$f_y(x,y)\Big|_{y=-3} = x\cdot(x-2)\cdot(2\cdot(-3)+3) = 0$$

$$\Rightarrow \quad -3x\cdot(x-2)=0$$

$$\Rightarrow \quad x=0 \quad \text{OR} \quad x=2$$

$\Rightarrow$ we have two critical points associated with this case given by

$\boxed{(0,-3)}$ AND $\boxed{(2,-3)}$.

Theorem 12.14. ***Second Partial Derivatives Test*** *p. 941*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. Suppose that $f(x, y)$ is twice differentiable on an open disk centered at the point (a, b) where $\nabla f(a, b) = 0$. Define the **discriminant** of f to be the function

$$D(x, y) = f_{xx}(x, y) \cdot f_{yy}(x, y) - (f_{xy}(x, y))^2 \tag{1}$$

Then, we can use this function to make the following conclusions:

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b)
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b)
3. If $D(a, b) < 0$, then f has a saddle point at (a, b)
4. If $D(a, b) = 0$, then this test is inconclusive and cannot be used to identify the behavior of f at point (a, b)

Definition. ***Saddle Point*** *p. 940*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. The function f is said to have a **saddle point** at the critical point (a, b) if and only if in every disk centered at (a, b):

1. there is at least one point (x, y) at which $f(x, y) > f(a, b)$
2. here is at least one (different) point (x, y) at which $f(x, y) < f(a, b)$

Example 12.8.3 p. 942) Use the second partial derivative test to classify all the critical points of

$$f(x,y) = x \cdot y \cdot (x-2) \cdot (y+3)$$

Solution: By our earlier work on example 12.8.1, we know we have 5 critical points given by

$(0,0)$ $(2,0)$ $(1,-3/2)$ $(0,-3)$ $(2,-3)$

At each of these points, we need to evaluate the discriminant

$$D(x,y) = f_{xx} f_{yy} - f_{xy}^2$$

and the value of f_{xx} to check the signs and apply our theorem.

Example 12.8.3 p. 942...

We begin by constructing $D(x, y)$

$$f(x,y) = x \cdot y\,(x-2)(y+3)$$

$$\Rightarrow \quad f_x(x,y) = 2y\,(y+3) \cdot (x-1)$$

$$\Rightarrow \quad f_{xx}(x,y) = 2y\,(y+3)$$

AND

$$\Rightarrow \quad f_y(x,y) = x \cdot (x-2)(2y+3)$$

$$\Rightarrow \quad f_{yy}(x,y) = 2x \cdot (x-2)$$

AND

$$\Rightarrow \quad f_{xy}(x,y) = \frac{\partial}{\partial y}\left[2(y^2+3y)\cdot(x-1)\right]$$

$$= 2\cdot(x-1)\cdot\frac{\partial}{\partial y}\left[y^2+3y\right]$$

$$= 2\cdot(x-1)\cdot(2y+3) = f_{yx}(x,y)$$

Then, we can construct our discriminant function

$$D(x,y) = \left[2y\cdot(y+3)\right]\cdot\left[2x\cdot(x-2)\right] - \left[2\cdot(x-1)\cdot(2y+3)\right]^2$$

use Mathematica to create a user defined function to simplify evaluation problem

Now we can create a table that includes the value of $D(x,y)$ and $f_{xx}(x,y)$ at each critical point

Critical Point	Discriminant $D(x,y)$	$f_{xx}(x,y)$	2nd partial Derivative test Conclusion
$(0,0)$	-36	0	Saddle Point
$(2,0)$	-36	0	Saddle Point
$(1,-3/2)$	9	$-\frac{9}{2}$	Local max
$(0,-3)$	-36	0	saddle point
$(2,-3)$	-36	0	Saddle Point

We can confirm this behaviour using Mma graphing capabilities including Plot3D and Countour Plot

Example 12.8.9 p. 948) Find the point(s) on the plane

$$x + 2y + z = 2$$

closest to the point $P(2, 0, 4)$.

Solution 1 : Use 2nd partial derivative test

Let (x, y, z) be any point on the plane. By definition, this point satisfies the equation of the plane and we have

$$z = 2 - x - 2y$$

We know that the distance between the point $P(2, 0, 4)$ and any point $(x, y, z) \in \mathbb{R}^3$ is given by function

$$d(x, y, z) = \sqrt{(x-2)^2 + (y-0)^2 + (z-4)^2}$$

If we require our point (x, y, z) to be on the plane, then we can create a two-variable distance function

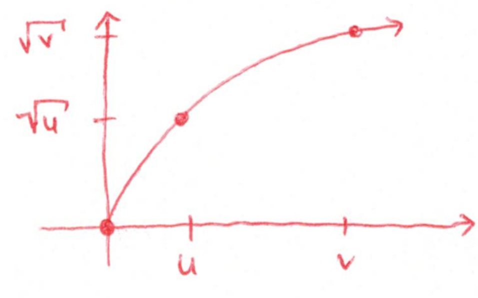
$$d(x,y) = \sqrt{(x-2)^2 + y^2 + (\underbrace{2 - x - 2y}_{\text{equation for } z \text{ that comes from substitution}} - 4)^2}$$

$$= \sqrt{(x-2)^2 + y^2 + (-x - 2y - 2)^2}$$

In this problem, we want to minimize $d(x,y)$. However, since $d(x,y)$ involves a square root, taking derivatives will be algebraically messy. Recall that $\sqrt{u} \le \sqrt{v} \iff u \le v$

The square root function is monotonically increasing

the order of output values is identical to order of input values



Thus, instead of trying to minimize $d(x,y)$, lets define a "new" function

$$f(x,y) = [d(x,y)]^2$$

whose minimum occurs at same point(s) (x,y).

Now, we want to find

$$\min_{(x,y)\in\mathbb{R}^2} f(x,y) = \min_{(x,y)\in\mathbb{R}^2} d^2(x,y)$$

$$= \min_{(x,y)\in\mathbb{R}^2} (x-2)^2 + y^2 + (x+2y+2)^2$$

this simplification comes from a side note using algebra.

$$= \min_{(x,y)\in\mathbb{R}^2} 2x^2 + 5y^2 + 4xy + 8y + 8$$

Now we can apply our second partial derivative test.

We begin by finding critical point(s) $\vec{\nabla} f = \vec{0}$:

$\boxed{\text{Eq I}}$ $f_x(x,y) = 4x + 4y = 0$

$\boxed{\text{Eq II}}$ $f_y(x,y) = 4x + 10y + 8 = 0$

By equation I, we know $x = -y$ and we substitute this into equation II to find

$$-4y + 10y + 8 = 0 \Rightarrow 6y = -8 \Rightarrow y = -4/3$$

Then, we have a single critical point at $\boxed{(4/3, -4/3)}$

Now, we test the discriminant and f_{xx} at this point with

$$D(x,y) = [f_{xx}] \cdot [f_{yy}] - [f_{xy}]^2$$

$$= [4] \cdot [10] - 4^2$$

$$= 40 - 16 = 24 > 0$$

$$f_{xx} = 4 > 0$$

These values are constant for all values of (x,y)

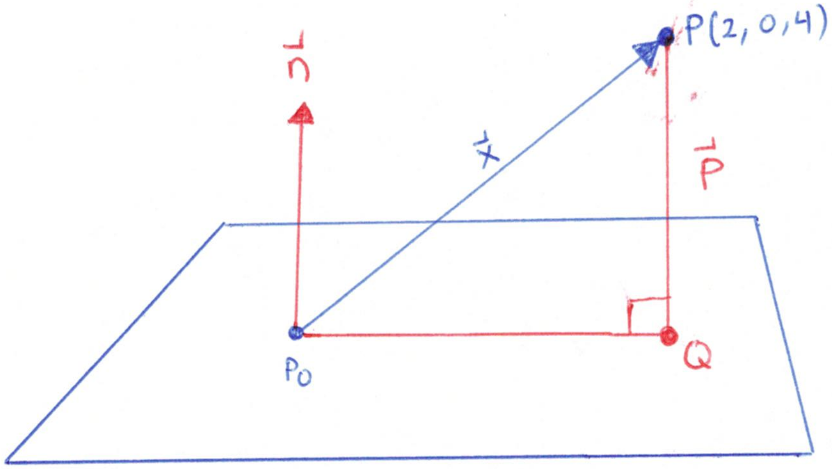
We see, then, point $(4/3, -4/3)$ is a local min of $f(x,y)$. Moreover, since $z = 2 - x - 2y$, we know our z-coordinate is

$$z = 2 - \frac{4}{3} + 2 \cdot \frac{4}{3} = 2 + \frac{8-4}{3} = \frac{10}{3}$$

and point $(4/3, -4/3, \frac{10}{3})$ is the closest on the given plane to point $P(2,0,4)$.

Solution 2: Use orthogonal projections

Consider the diagram below



Plane : $x + 2y + z = 2$

Let $\vec{n} = \langle 1, 2, 1 \rangle$ be the normal vector to our plane. Let P_0 be any point on our plane say the point

$P_0(0,1,0)$ ← This point satisfies the equation for the plane since

$0 + 2 \cdot 1 + 0 = 2$ ✓

Define $\vec{x} = \overrightarrow{P_0P} = \langle 2, -1, 4 \rangle$ be the vector connecting P_0 to P.

Then we can define $\vec{p} = \text{Proj}_{\vec{n}}(\vec{x})$

the orthogonal projection of $\vec{x}$ onto $\vec{n}$. We

Know then

$\vec{n} = \langle 1, 2, 1 \rangle$

$$\vec{p} = \frac{\vec{x} \cdot \vec{n}}{\|\vec{n}\|_2} \cdot \frac{\vec{n}}{\|\vec{n}\|}$$

$\Rightarrow \|\vec{n}\|_2 = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$

$$= \frac{\langle 2, -1, 4 \rangle \cdot \langle 1, 2, 1 \rangle}{\sqrt{6}} \cdot \frac{\langle 1, 2, 1 \rangle}{\sqrt{6}}$$

$$= \left[\frac{2 - 2 + 4}{6} \right] \cdot \langle 1, 2, 1 \rangle$$

$$= \langle \tfrac{2}{3}, \tfrac{4}{3}, \tfrac{2}{3} \rangle$$

Notice, since this scalar is positive, we Know $\vec{p}$ is oriented in the same direction as $\vec{n}$

If we set point $Q(x, y, z)$ to be point on the plane "directly under" point P with

$$\vec{p} = \overrightarrow{QP}$$

then, we can find the coordinates of Q since

$$\overrightarrow{QP} = \langle 2 - x, 0 - y, 4 - z \rangle = \langle \tfrac{2}{3}, \tfrac{4}{3}, \tfrac{2}{3} \rangle$$

$$\Rightarrow \quad 2 - x = \frac{2}{3}$$

$$-y = 4/3$$

$$4 - z = \frac{2}{3}$$

$$\Rightarrow \quad x = 2 - 2/3 = 4/3$$

$$y = -4/3$$

$$z = 4 - 2/3 = 10/3$$

$$\Rightarrow \boxed{Q(4/3, -4/3, 10/3)}$$

$\boxed{\text{LB, P21}}$