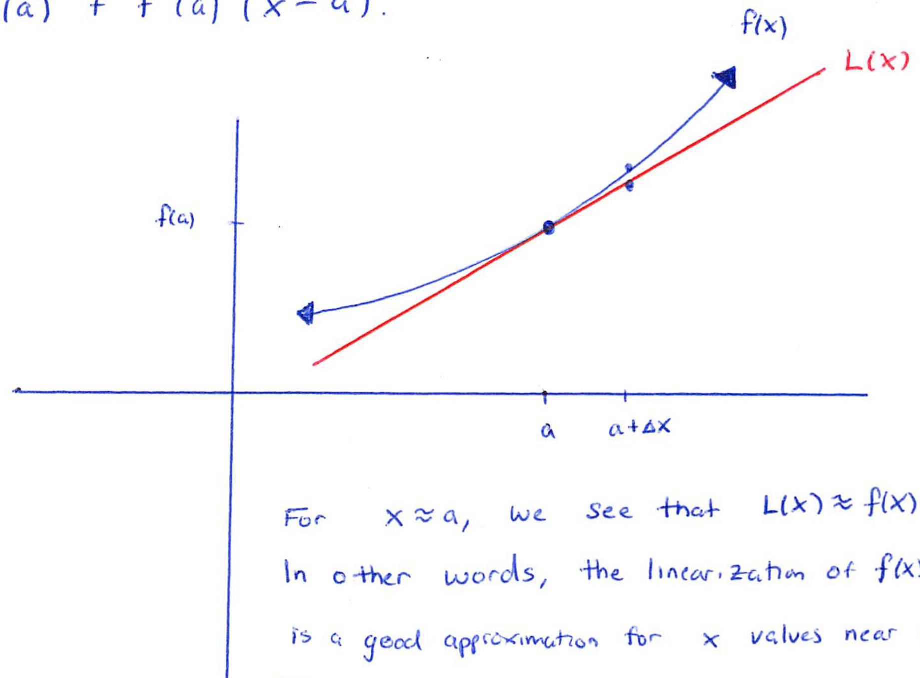
Lesson 12: Tangent Planes

Linearization and Approximations (Section 12.7 p.928)

For function $f(x)$, we defined the linearization of f at point $(a, f(a))$ as follows

$$L(x) = f(a) + f'(a)(x-a).$$

Recall:

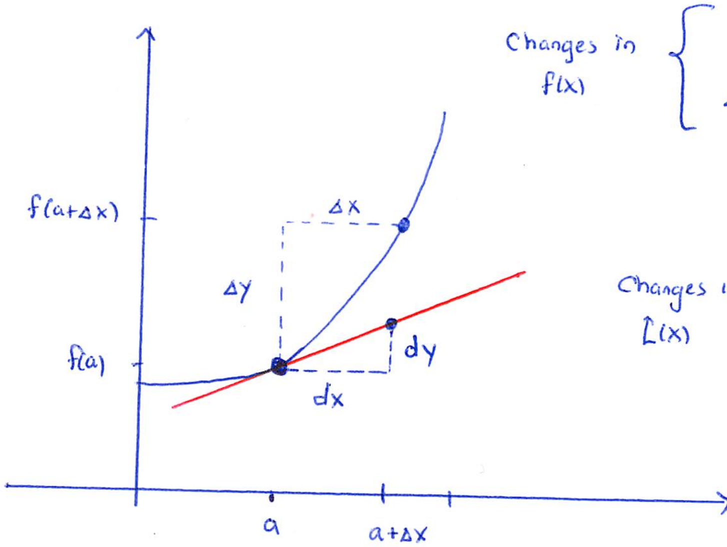


For $x \approx a$, we see that $L(x) \approx f(x)$.

In other words, the linearization of $f(x)$ is a good approximation for x values near a.

(The more we zoom in on a point on a smooth curve, the more the tangent line looks like the curve).

We also described this termonology in terms of differentials. Let's exagerate for effect:



Changes in $f(x)$ $\begin{cases} \Delta x = x + \Delta x - x = \Delta x \\ \Delta y = f(x+\Delta x) - f(x) \end{cases}$

Changes in $L(x)$ $\begin{cases} dx = x + \Delta x - x = \Delta x \\ dy = L(x+\Delta x) - L(x) \end{cases}$

$= f'(x)\,dx$

Show how to derive this

$$\boxed{L(a+\Delta x) = f(a) + dy, \quad dy = f'(a)\,dx}$$

For $x \approx a$, we have $\Delta y \approx dy$ and the differential dy is a good approximation for change along function.

Now that we are working with multivariable functions, let's develop an analogous idea in three dimensions: tangent plane approximations to our surface at a point.

Tangent Planes for Implicitly Defined Surface $F(x,y,z) = 0$

Recall that many surfaces can be defined in the form

$$F(x,y,z) = 0$$

This was called a level surface associated with our function $F(x,y,z)$ (all input points that produce output 0).

Where $F(x,y,z)$ is differentiable.

Tangent Plane to Level Surface $F(x,y,z) = 0$

Suppose we have a surface in $\mathbb{R}^3$ that is defined implicitly by equation $F(x,y,z) = 0$. We want to find an equation for a plane tangent to this surface at a given point (a,b,c).

To this end, suppose we have a smooth curve C on our surface, given by parametric equation

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

Because this curve lies on our surface, each point on C satisfies the surface equation

$$\underbrace{F(x(t), y(t), z(t))} = 0$$

Function value evaluated at any point on curve C

Now, if we differentiate both sides of this equation with respect to t, we find a very useful relationship that extends Theorem 12.12 p. 922 to level surfaces, with

$$\frac{d}{dt}\left[F(x(t), y(t), z(t))\right] = \frac{d}{dt}[0]$$

$$\Rightarrow \quad \frac{\partial F}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial F}{\partial z}\cdot\frac{dz}{dt} = 0$$

$$\Rightarrow \quad \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = 0$$

$$\Rightarrow \quad \vec{\nabla} F(x,y,z) \cdot \vec{r}\,'(t) = 0$$

Then, the tangent vector $\vec{r}\,'(t)$ to curve C at any point is orthogonal to the gradient $\vec{\nabla}F$ at that point.

Now, let's fix a specific point on our surface

$$P_0(a, b, c)$$

and assume $\vec{\nabla} F(a,b,c) \neq \vec{0}$. If we consider all possible smooth curves C passing through point P_0, we know that any vector tangent to C is orthogonal to $\vec{\nabla} F(a,b,c)$. This argument holds for all possible curves C and thus all of the tangent vectors to these curves lie in a single plane, called the tangent plane. Moreover, $\vec{\nabla} F(a,b,c)$ is the normal vector to this plane. Since we have a known point on our plane and a normal vector we can write an equation for this plane:

$$0 = \vec{\nabla} F(a,b,c) \cdot \langle x-a,\ y-b,\ z-c \rangle$$

Definition. *p. 939* ***Equation of the Tangent Plane for surface*** $F(x, y, z) = 0$

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a three-variable function. Suppose that F is differentiable at the point $P_0(a, b, c)$ with $\nabla F(a, b, c) \neq \mathbf{0}$. The **tangent plane** to the level surface $F(x, y, z) = 0$ at P_0 is the plane passing through P_0 and orthogonal to $\nabla F(a, b, c)$. The equation for this plane is

$$0 = \nabla F(a, b, c) \cdot \langle x - a,\ y - b,\ z - c \rangle$$

$$= F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c)$$

Theorem. *p. 931* ***Tangent Plane for surface*** $z = f(x, y)$

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function that is differentiable at the point (a, b). An equation of the plane tangent to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Example 12.7.1 p.930)

For the ellipsoid defined by the equation

$$F(x,y,z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0$$

find the equation for the tangent plane at point $(0, 4, 3/5)$.

Also, find any point(s) on the ellipsoid with a horizontal tangent plane.

Solution: We know that our desired tangent plane has equation

$$0 = \vec{\nabla} F(0, 4, 3/5) \cdot \langle x, y-4, z-3/5 \rangle$$

We also know our gradient function is given by

$$\vec{\nabla} F(x,y,z) = \left\langle \frac{2x}{9}, \frac{2y}{25}, 2z \right\rangle$$

$$\Rightarrow \vec{\nabla} F(0, 4, 3/5) = \langle 0, 8/25, 6/5 \rangle$$

Then, our desired tangent plane is given by

$$\frac{8}{25}(y-4) + \frac{6}{5}(z - 3/5) = 0$$

Example 12.7.1 p. 930 ...)

To find the point(s) on the surface with horizontal tangent plane, we note that our normal vector should have form $\vec{n} = \langle 0, 0, c \rangle$ for some $c \in \mathbb{R}$.

(since horizontal planes are parallel to the xy-plane).

Then we want to find points on our surface such that

$$\vec{\nabla} F(x,y,z) = \left\langle \frac{2x}{9}, \frac{2y}{25}, 2z \right\rangle = \langle 0, 0, c \rangle$$

In other words, we require

$$F_x(x,y,z) = \frac{2x}{9} = 0 \quad \Rightarrow \quad x = 0$$

$$F_y(x,y,z) = \frac{2y}{25} = 0 \quad \Rightarrow \quad y = 0$$

We see there are two points on the plane with

$z^2 = 1 \Rightarrow (0, 0, 1)$ and $(0, 0, -1)$.

Tangent Planes for Surface $z = f(x,y)$

Surfaces in $\mathbb{R}^3$ that are defined explicitly as

$$z = f(x,y)$$

can be viewed as a special case of a level surface defined by $F(x,y,z) = 0$.

In particular,

$$z = f(x,y) \Rightarrow z - f(x,y) = 0$$

$$\Rightarrow F(x,y,z) = 0$$

for $F(x,y,z) = z - f(x,y)$.

To find the tangent plane to this surface, we calculate the gradient

$$\vec{\nabla} F(x,y,z) = \left\langle \frac{\partial}{\partial x}[F], \frac{\partial}{\partial y}[F], \frac{\partial}{\partial z}[F] \right\rangle$$

$$= \langle -f_x(x,y), -f_y(x,y), 1 \rangle$$

We can now define our equation to our tangent plane to $z = f(x,y)$ at point $(a, b, f(a,b))$ as

$$0 = \vec{\nabla} F(a,b,f(a,b)) \cdot \langle x-a,\ y-b,\ z-f(a,b) \rangle$$

$$= -f_x(a,b)\cdot(x-a) - f_y(a,b)\cdot(y-b) + z - f(a,b)$$

We can use algebra to solve for z in terms of x and y.

Suppose $f(x,y)$ has continuous partial derivatives. An equation ~~to~~ of the tangent plane to the surface $z = f(x,y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Similar to
Example 12.7.2 p. 931

Let's find the equation of the tangent plane to the elliptic paraboloid

$$z = f(x,y) = 2x^2 + y^2$$

at the point $(1,1,3)$

Solution: We know

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

gives our tangent plane. In this case we have

$$(x_0, y_0, z_0) = (1, 1, 3) \qquad \begin{cases} x_0 = 1 \\ y_0 = 1 \\ z_0 = 1 \end{cases}$$

Using our knowledge of partial derivatives, we see

$$f_x(x,y) = \frac{\partial}{\partial x}\left[2x^2 + y^2\right] = 4x$$

$$\Rightarrow f_x(x_0, y_0) = f_x(1,1) = 4 \cdot 1 = 4$$

Similar to

Example 12.7.2 p.931 continued...

Similarly, we have

$$f_y(x,y) = \frac{\partial}{\partial y}\left[2x^2 + y^2\right] = 2y$$

$$\Rightarrow \quad f_y(x_0,y_0) = f_y(1,1) = 2\cdot 1 = 2$$

Then, the equation of our tangent plane is given by

$$z - 3 = 4(x-1) + 2(y-1)$$

$$\Rightarrow \quad z = 3 + 4x - 4 + 2y - 2$$

$$\Rightarrow \quad \boxed{z = 4x + 2y - 3}$$

Definition. *p. 931* ***Linear Approximation***

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function that is differentiable at the point (a, b). The **linear approximation** to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane to the surface at that point, given by the equation

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Definition. *p. 933* ***The differential** dz*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function that is differentiable at the point (a, b). The change in output values of $z = f(x, y)$ as the input variables change from (a, b) to $(a + dx, b + dy)$ is denoted as Δz and is approximated by the **differential**

$$\Delta z \approx dz = f_x(a, b)\, dx + f_y(a, b)\, dy$$

Example 12.7.3 p. 932)

Let $f(x,y) = \dfrac{5}{x^2+y^2}$. Find the linear approximation to f at point $(-1, 2, 1)$. Use this linearization to approximate $f(-1.05, 2.1)$.

Solution: We recall that the linear approximation to our surface $z = f(x,y)$ at point $(a, b, f(a,b))$ is

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

In this case, we have $(a,b) = (-1, 2)$ and we verify that

$$f(-1,2) = \frac{5}{(-1)^2+2^2} = \frac{5}{1+4} = 1 \checkmark$$

We see also that

$$f_x(-1,2) = \left.\frac{-10x}{(x^2+y^2)^2}\right|_{(-1,2)} = \frac{10}{25} = \frac{2}{5}$$

$$f_y(-1,2) = \left.\frac{-10y}{(x^2+y^2)^2}\right|_{(-1,2)} = \frac{-20}{25} = \frac{-4}{5}$$

Then, our linearization of $f(x,y)$ at this point is given by

$$L(x,y) = 1 + \frac{2}{5}(x+1) - \frac{4}{5}(y-2)$$

We can use this linear approximation to estimate our function value at input close to $(-1,2)$. In particular, we see

$$f(-1.05, 2.1) \approx L(-1.05, 2.1)$$

$$= 1 + \frac{2}{5}(-1.05+1) + \frac{-4}{5}(2.1-2)$$

$$= 1 + \frac{2}{5} \cdot \frac{-5}{100} + \frac{-4}{5} \cdot \frac{1}{10}$$

$$= 1 + \frac{-2}{100} - \frac{4}{50}$$

$$= 1 + \frac{-5}{50}$$

$$= 1 - \frac{1}{10} \neq \boxed{0.90}$$

Similar to

Exercise 12.7.31 p.936

Let $z = f(x,y) = x^2 + 3xy - y^2$.

(i) Find differential dz

(ii) If changes from 2 to 2.05 and

y changes from 3 to 2.96, compare Δz & dz

Solution: (i) Recall from our previous discussion we have

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

In this case,

$$f_x(x,y) = \frac{\partial}{\partial x}\left[x^2 + 3xy - y^2\right] = 2x + 3y$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left[x^2 + 3xy - y^2\right] = 3x - 2y$$

$$\Rightarrow \boxed{dz = (2x + 3y)\,dx + (3x - 2y)\,dy}$$

(ii) For our given values: $dx = 2.05 - 2 = 0.05$, $x = 2$

$dy = 2.96 - 3 = -0.04$, $y = 3$

$$\Rightarrow dz = [2\cdot 2 + 3\cdot 3]\,0.05 + [3\cdot 2 - 2\cdot 3](-0.04) \doteq \boxed{0.65}$$

On the other hand, $\Delta z = f(2.05, 2.96) - f(2,3) \doteq \boxed{0.6449}$

Similar to

Exercise 12.7.33 p.936 Let $f(x,y) = xe^{xy}$.

i. Show $f(x,y)$ is differentrable at $(1,0)$

ii. Find the linearization of $f(x,y)$ at $(1,0)$

iii. Approximate $f(x,y)$ at $(1.1, -0.1)$

Solution: i. We begin by finding the partial derivatives of $f(x,y)$

$$f_x(x,y) = \frac{\partial}{\partial x}\left[x \cdot e^{xy}\right] = e^{xy} + xye^{xy}$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left[x \cdot e^{xy}\right] = x^2 e^{xy}$$

Since both f_x and f_y are continuous, f is differentiable by theorem 8.

ii. $$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$$

$$= 1 + 1(x-1) + 1(y-0)$$

$$= 1 + x - 1 + y$$

$$= \boxed{x+y}$$

iii. $f(x,y) = xe^{xy} \approx x+y$ near $(1,0)$

$\Rightarrow f(1.1, -0.1) \approx 1.1 - 0.1 = 1$

Compare w/ actual value $f(1.1, -0.1) = 0.98542$