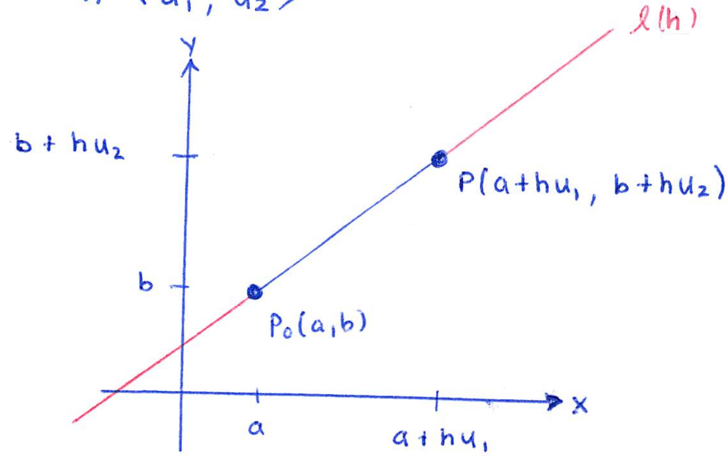
Lesson 11: Directional Derivatives and the Gradient

Limit Definition of Directional Derivative

Let $z = f(x,y)$ define a surface, where $f(x,y)$ is differentiable. Let point $(a, b, f(a,b))$ be on the surface. Let $\vec{u} = \langle u_1, u_2 \rangle \in \mathbb{R}^2$ be a unit vector in the xy-plane.

Suppose we move along the line $\vec{\ell}(h)$ is the domain, where

$$\vec{\ell}(h) = \langle a, b \rangle + h \langle u_1, u_2 \rangle$$



$$= \vec{P_0} + h \cdot \vec{u}$$

← recall: this is the line through point $\vec{P_0}(a,b)$ in the direction of vector $\vec{u}$.

$$= \langle a + hu_1, \ b + hu_2 \rangle = \langle x(h), y(h) \rangle$$

Notice: in this parametric equation for our line $\vec{\ell}(h)$, we are using parameter h to represent the "amount" of vector $\vec{u}$ we move away from point $\vec{p_0}$.

If we move along $\vec{\ell}(h)$ in the domain of $f(x,y)$, we can trace a curve C along the surface $z = f(x,y)$ where the output points of C are given by

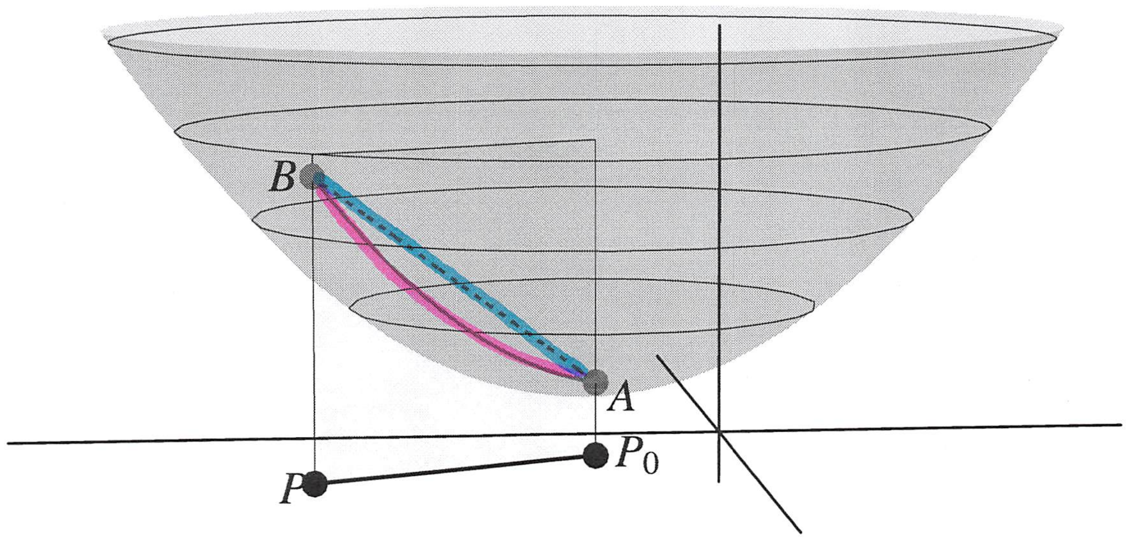
$$g(h) = f(\vec{\ell}(h)) = f(a + hu_1 \, , \, b + hu_2)$$

The curve C described above can also be visualized as the intersection of a plane through point $P_0(a,b,0)$ with normal vector $\langle -u_2 \, , u_1 \, , 0 \rangle = \vec{n}$ and the surface $z = f(x,y)$. Notice this plane is perpendicular to the xy-plane and contains the line $\vec{\ell}(h)$.

Point A: $(a, b, g(0)) = (a, b, f(a,b))$

Point B: $(a+u_1 \cdot h,\ b+u_2 \cdot h,\ g(h))$

where $g(h) = f(a+h\cdot u_1,\ b+h\cdot u_2)$



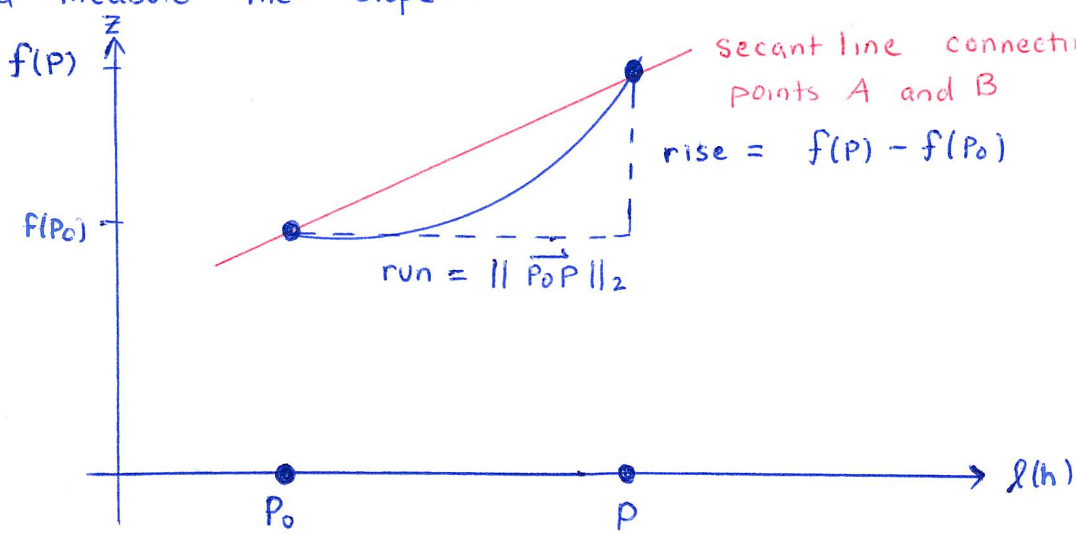
Here, the curve C given by $g(h)$ is highlighted in pink. Notice we can draw a secant line through points A and B on this curve, highlighted in blue. The slope of this secant line is given as

$$m_{AB} = \frac{\text{rise}}{\text{run}} = \frac{g(h) - g(0)}{\| \overrightarrow{P_0P} \|_2}$$

$$= \frac{f(a+h\cdot u_1,\ b+h\cdot u_2) - f(a,b)}{\| \overrightarrow{P_0P} \|_2}$$

Then, let's define points $A(a, b, f(a,b))$ and $B(a+hu_1, b+hu_2, f(a+hu_1, b+hu_2))$. Notice, these points sit on curve C with A coming from $h=0$ while B results from finding $g(h)$.

We can draw a secant line through points A and B and measure the slope of this line

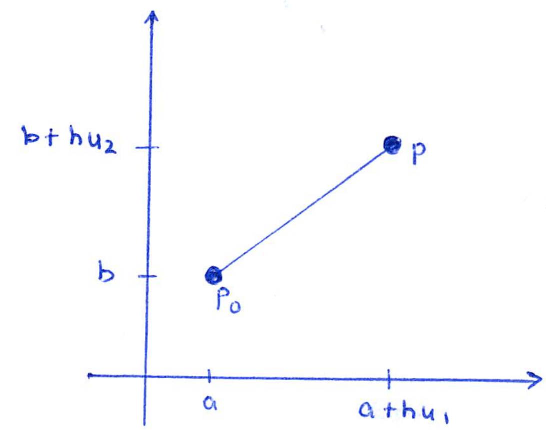


$$\text{Slope of secant line} = \frac{\text{rise}}{\text{run}}$$

$$= \frac{f(P) - f(P_0)}{\|\overrightarrow{P_0P}\|_2}$$

$$= \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

see next page to understand why run $= h$.



In order to calculate the run (the change in distance in the domain space), we want to find the length of the vector $\overrightarrow{P_0P}$ connecting point $P_0(a, b)$ to point $P(a+hu_1, b+hu_2)$.

To this end, notice

$$\begin{aligned}\overrightarrow{P_0P} &= \langle a + hu_1 - a,\ b + hu_2 - b\rangle \\ &= \langle hu_1,\ hu_2\rangle \\ &= h\cdot\langle u_1, u_2\rangle \\ &= h\,\vec{u}\end{aligned}$$

Since $\|\vec{u}\| = 1$ by assumption, we travel exactly h units to get from P_0 to P. Thus, our run is simply h.

In order to find the slope of the tangent line to the curve C at the point A, we take the limit

$$\lim_{P \to P_0} \frac{f(P) - f(P_0)}{\|\overrightarrow{P_0P}\|_2} = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

When this limit exists, it is called the directional derivative of f at (a,b) in the direction of $\vec{u}$.

The derivation of the slope of the tangent line to C at point A gives rise to the formal, limit-based definition for the directional derivative. This definition helps illustrate the connection to our understanding of derivatives as the limit of a slope of the secant line.
However, when calculating directional derivatives, we prefer to avoid formal limits (as these can be quite cumbersome and time consuming).

Dot Product Formula for Directional Derivative.

To avoid using the limit while calculating directional derivatives, let's define the single variable function

$$g(t) = f(\vec{\ell}(t)) \qquad \text{where } \vec{\ell}(t) = \vec{P}_0 + t\vec{u}$$

$$= \langle a + tu_1, b + tu_2 \rangle$$

$$= \langle x(t), y(t) \rangle$$

where the output of $g(t)$ is the z-value on surface along the line $\vec{\ell}(t)$ in the domain of $f(x,y)$.

Notice that $g'(t) = \frac{d}{dt}[g(t)]$ is the derivative of f along the line $\vec{\ell}(t)$ (the slope of the tangent line to curve C at point $P(a+tu_1, b+tu_2) = \vec{\ell}(t)$).

Then, the directional derivative of f at (a,b) in the direction of $\vec{u}$ is alternatively

$$D_{\vec{u}} f(a,b) = g'(0)$$

recall $\vec{\ell}(0) = (a,b)$

$$= \frac{d}{dt}\left[g(t) \right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left[f(\vec{\ell}(t)) \right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left[f(x(t), y(t)) \right]\Big|_{t=0}$$

$$= \left[\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \right]\Big|_{t=0}$$

by multivariable chain rule

$$= f_x(a,b) \cdot u_1 + f_y(a,b) \cdot u_2$$

$$\left\{ \begin{array}{l} \frac{dx}{dt} = \frac{d}{dt}\left[a + tu_1\right] = u_1 \\ \frac{dy}{dt} = \frac{d}{dt}\left[b + tu_2\right] = u_2 \end{array} \right.$$

$$= \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle$$

Now we can compute the directional derivative as a dot product, which provides a practical formula to find $D_{\vec{u}} f$.

Definition. *p. 917* ***Limit Definition of the Directional Derivative***

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function that is differentiable at point (a, b). Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the $xy-$plane. The **directional derivative of f at (a, b) in the direction of $\mathbf{u}$ is**

$$D_{\mathbf{u}}f(a,b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}$$

provided this limit exists.

Theorem 12.10. *p. 918* ***Dot Product Formula for the Directional Derivative***

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function that is differentiable at point (a, b). Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the $xy-$plane. The **directional derivative of f at (a, b) in the direction of $\mathbf{u}$ is**

$$D_{\mathbf{u}}f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle = \vec{\nabla} f(a,b) \cdot \mathbf{u}$$

Example 12.6.1 p. 918)

Consider the paraboloid

$$z = f(x,y) = \frac{1}{4}(x^2 + 2y^2) + 2$$

Let $P_0(3,2)$ and define unit vectors

$$\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \quad \text{and} \quad \vec{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$$

Find $D_{\vec{u}} f(3,2)$ and $D_{\vec{v}} f(3,2)$.

☐ we should check for ourselves that these are unit vectors

Solution: By theorem 12.10 p. 918, we know that at any point $P(a,b)$ and for any unit vector $\vec{u} = \langle u_1, u_2 \rangle$, we have

$$D_{\vec{u}} f(a,b) = f_x(a,b) \cdot u_1 + f_y(a,b) \cdot u_2$$

$$= \underbrace{\langle f_x(a,b), f_y(a,b) \rangle}_{\text{we call this the gradient vector } \vec{\nabla} f(a,b)} \cdot \langle u_1, u_2 \rangle$$

In this case, we see

$$f_x(x,y) = \frac{\partial}{\partial x}\left[\frac{1}{4}(x^2+2y^2) + 2\right] = \frac{x}{2}$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left[\frac{1}{4}(x^2+2y^2) + 2\right] = y$$

Example 12.6.1 p. 918...)

Then $f_x(3,2) = 3/2$ and $f_y(3,2) = 2$.

With this, we can calculate

$$D_{\vec{u}} f(3,2) = \langle f_x(3,2), f_y(3,2) \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

$$= \langle \frac{3}{2}, 2 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

$$= \frac{3}{2} \cdot \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}}$$

$$= \boxed{\frac{7}{2\sqrt{2}}}$$

Similarly, we have

$$D_{\vec{v}} f(3,2) = \langle f_x(3,2), f_y(3,2) \rangle \cdot \langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \rangle$$

$$= \langle \frac{3}{2}, 2 \rangle \cdot \langle \frac{1}{2}, \frac{-\sqrt{3}}{2} \rangle$$

$$= \frac{3}{2} \cdot \frac{1}{2} + 2 \cdot \frac{-\sqrt{3}}{2}$$

$$= \boxed{\frac{3}{4} - \sqrt{3}.}$$

Definition. *p. 919* ***Gradient (in Two Dimensions)*** *p. 919*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function, differentiable at point (x, y). The **gradient** of f at (x, y) is the vector-valued function

$$\vec{\nabla} f(x,y) = \langle\ f_x(x,y),\ f_y(x,y)\ \rangle = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j}$$

Definition. *p. 924* ***Gradient (in Three Dimensions)*** *p. 919*

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a three-variable function, differentiable at point (x, y). The **gradient** of f at (x, y, z) is the vector-valued function

$$\vec{\nabla} f(x,y) = \langle\ f_x(x,y,z),\ f_y(x,y,z),\ f_z(x,y,z)\ \rangle$$

$$= f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k}$$

Example 12.6.3 p. 919) Let $f(x,y) = 3 - \frac{x^2}{10} + \frac{xy^2}{10}$. Find

$\vec{\nabla} f(3,-1)$ and $D_{\vec{u}} f(3,-1)$ where $\vec{u} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

Solution: By definition, $\vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$. In this case, we see

$$f_x(x,y) = \frac{\partial}{\partial x}\left[3 - \frac{x^2}{10} + \frac{xy^2}{10}\right] = -\frac{x}{5} + \frac{y^2}{10}$$

$$f_y(x,y) = \frac{\partial}{\partial y}\left[3 - \frac{x^2}{10} + \frac{xy^2}{10}\right] = \frac{xy}{5}$$

Then, we can find our desired gradient vector

$$\vec{\nabla} f(3,-1) = \langle \frac{-3}{5} + \frac{1}{10}, \frac{-3}{5} \rangle = \langle -\frac{1}{2}, -\frac{3}{5} \rangle$$

To find the directional derivative of f at $(3,-1)$ in the direction of $\vec{u}$, we first check $\|\vec{u}\|_2 = 1$.

Get into habit of checking $\vec{u}$ is unit vector.

$\|\vec{u}\|_2^2 = \left[\frac{1}{\sqrt{2}}\right]^2 + \left[\frac{-1}{\sqrt{2}}\right]^2 = 1$ ✓

Now, we have

$$D_{\vec{u}} f(3,-1) = \vec{\nabla} f(3,-1) \cdot \vec{u} = \langle -\frac{1}{2}, -\frac{3}{5} \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle$$

$$= \frac{-1}{2\sqrt{2}} + \frac{3}{5\sqrt{2}} = \boxed{\frac{1}{10\sqrt{2}}}$$

The Gradient and Steepest Ascent and Descent

We will now develop some geometric intuition about the gradient as a vector $\vec{\nabla} f$.

Suppose we have a two-variable function $z = f(x,y)$ that is differentiable at point (a,b). We can calculate the directional derivative of f at (a,b) in the direction of any unit vector $\vec{u}$

$$D_{\vec{u}} f(a,b) = \vec{\nabla} f(a,b) \cdot \vec{u}$$

$$= \| \vec{\nabla} f(a,b) \|_2 \cdot \| \vec{u} \|_2 \cdot \cos(\theta)$$

by the cosine formula for the dot product where θ is the angle between $\vec{\nabla} f$ and $\vec{u}$

$$= \| \vec{\nabla} f(a,b) \|_2 \; \cos(\theta)$$

since $\|\vec{u}\|_2 = 1$ ($\vec{u}$ is a unit vector).

Notice that $-1 \leq \cos(\theta) \leq 1$ for all values of $\theta \in [0, 2\pi]$.

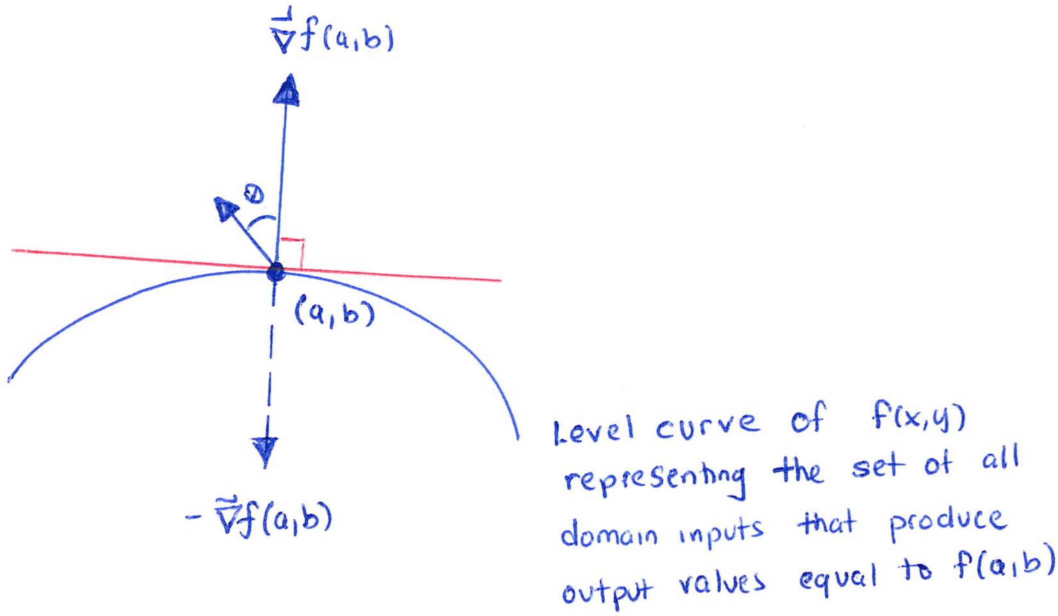
Steepest Ascent

From this analysis, we see $D_{\vec{u}} f(a,b)$ is maximum (with respect to angle θ) if $\cos(\theta) = 1$ which corresponds to $\theta = 0$.

In other words, the directional derivative has the steepest slope and f has greatest increase when $\vec{\nabla} f(a,b)$ and $\vec{u}$ point in the same direction. In this case $D_{\vec{u}} f(a,b) = \| \vec{\nabla} f(a,b) \|_2$

Steepest Descent

On the other hand, $D_{\vec{u}} f(a,b)$ is minimum if $\cos(\theta) = -1$ corresponding to $\theta = \pi$. Thus if $\vec{u}$ points in the opposite direction of $\vec{\nabla} f(a,b)$, this results in the greatest rate of decrease of f where $D_{\vec{u}} f(a,b) = -\| \vec{\nabla} f(a,b) \|_2$.



Finally, we notice that $D_{\vec{u}} f(a,b) = 0$ when the angle θ between $\vec{\nabla} f(a,b)$ and $\vec{u}$ is $\theta = \pi/2$. In this case, we move along the level curve C.

These observations result in Theorem 12.11 p. 920

Example 12.6.4 p. 920)

Consider the bowl-shaped paraboloid $z = f(x,y) = 4 + x^2 + 3y^2$.

At point $(2, -1/2, 35/4)$, what are the directions of maximal ascent and descent? What direction(s) result in no change in the function value?

Solution: At the point $(2, -1/2)$, the value of the gradient is

$$\vec{\nabla} f(2, -1/2) = \langle 2x, 6y \rangle \Big|_{(2,-1/2)} = \langle 4, -3 \rangle$$

Then, the direction of steepest ascent is

$$\vec{u} = \frac{1}{5} \langle 4, -3 \rangle$$

resulting in the greatest increase in values of f.

The direction of steepest descent is

$$-\vec{u} = -\vec{\nabla} f(2, -1/2) = \frac{1}{5} \langle -4, 3 \rangle .$$

Finally, at point $(2, -1/2)$ the function has no change when we move orthogonally to $\vec{\nabla} f$, which in this case is in direction

$$\vec{v} = \frac{1}{5} \langle 3, 4 \rangle \quad \text{or} \quad -\vec{v} = \frac{-1}{5} \langle 3, 4 \rangle$$

Theorem 12.11. *p. 920* ***The Gradient and Directions of Change***

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function that is differentiable at point (a, b), with $\nabla f(a, b) \neq \mathbf{0}$.

1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $\|\nabla f(a, b)\|_2$

2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-\|\nabla f(a, b)\|_2$

3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$

Theorem 12.12. *p. 922* ***The Gradient and Level Curves***

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function that is differentiable at point (a, b). Then, the tangent line to the level curve of f at (a, b) is orthogonal to the gradient vector $\nabla f(a, b)$, provided that $\nabla f(a, b) \neq \mathbf{0}$.

Example 12.6.6 p.922)

consider the upper sheet of the hyperboloid of two sheets

$$z = f(x,y) = \sqrt{1 + 2x^2 + y^2}$$

Find an equation of the tangent line to the level curve at (1,1) and verify that $\vec{\nabla} f(1,1)$ is orthogonal to this line.

Solution: we notice that $f(1,1) = \sqrt{1+2+1} = 2$. Then, our level curve is the ellipse given by equation

$$\sqrt{1 + 2x^2 + y^2} = 2$$

$$\Rightarrow \quad 1 + 2x^2 + y^2 = 4$$

$$\Rightarrow \quad 2x^2 + y^2 = 3$$

To find the equation for our tangent line, we need to find slope, in which case we using implicit differentraition

$$\frac{d}{dx}\left[2x^2 + y^2\right] = \frac{d}{dx}[3]$$

$$\Rightarrow \quad 4x + 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{y}$$

Thus, at point (1,1) we have tangent line with slope

$$\left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{2}{1} = -2$$

Our point slope form of our line is given by

$$y - 1 = -2\cdot(x-1)$$

$$\Rightarrow \quad y = -2x + 3$$

we can write this in vector form as

$$\vec{r}(t) = \langle t, -2t\rangle + \langle 0, 3\rangle$$

$$= \langle 0, 3\rangle + t\,\underbrace{\langle 1, -2\rangle}$$

$$= \vec{r}_0 + t\cdot\vec{v}$$

this is the direction of the tangent line

Now, we see $\vec{\nabla} f(x,y) = \left\langle \frac{2x}{\sqrt{1+2x^2+y^2}}, \frac{y}{\sqrt{1+2x^2+y^2}} \right\rangle$

$$\Rightarrow \vec{\nabla} f(1,1) = \langle 1, 1/2\rangle$$

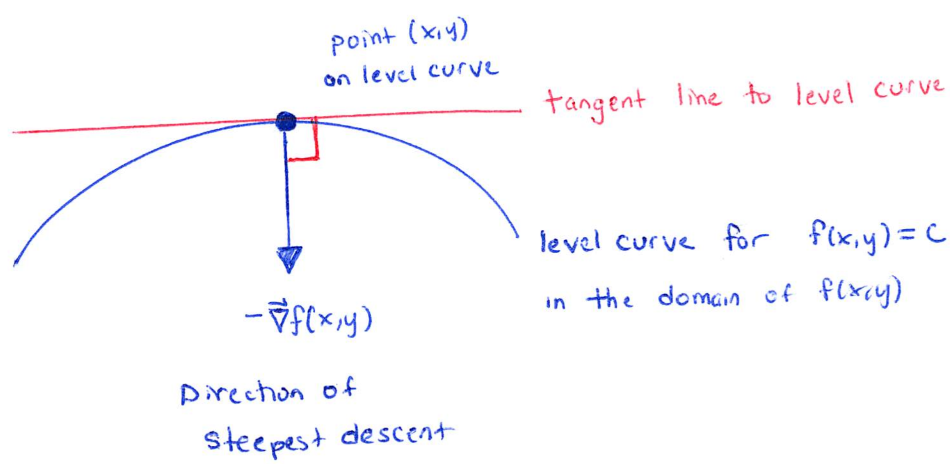
$$\Rightarrow \vec{\nabla} f(1,1)\cdot\vec{v} = \langle 1, 1/2\rangle\cdot\langle 1, -2\rangle = 0 \checkmark$$

Thus $\vec{\nabla} f$ is orthogonal to the tangent line to our level curve at this point, as we claimed.

Example 12.6.7 p.923

Consider the paraboloid $z = f(x,y) = 4 + x^2 + 3y^2$. If we begin at point $(3, 4, 61)$ on this surface, let's find the projection in the xy-plane of the path of steepest descent on the surface.

Solution: If we consider that path of steepest decent, we know that at any point $(x, y, f(x,y))$ on the surface, the pre image of this path should be orthogonal to the tangent line of the level curve at this point



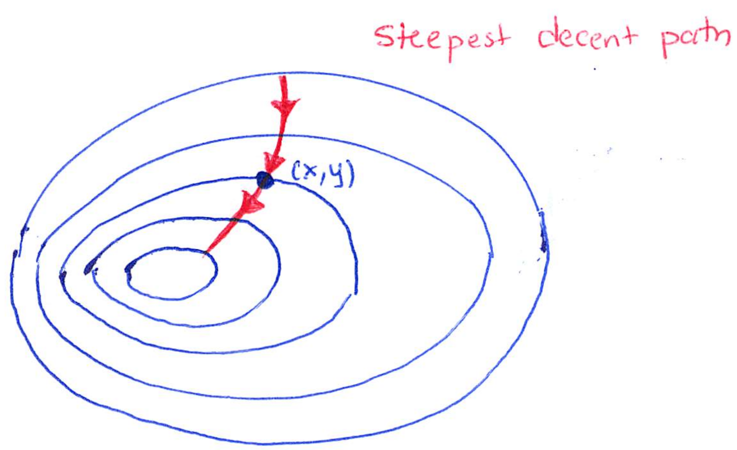
In this case, we can calculate

$$-\vec{\nabla} f(x,y) = -\langle f_x(x,y), f_y(x,y) \rangle$$

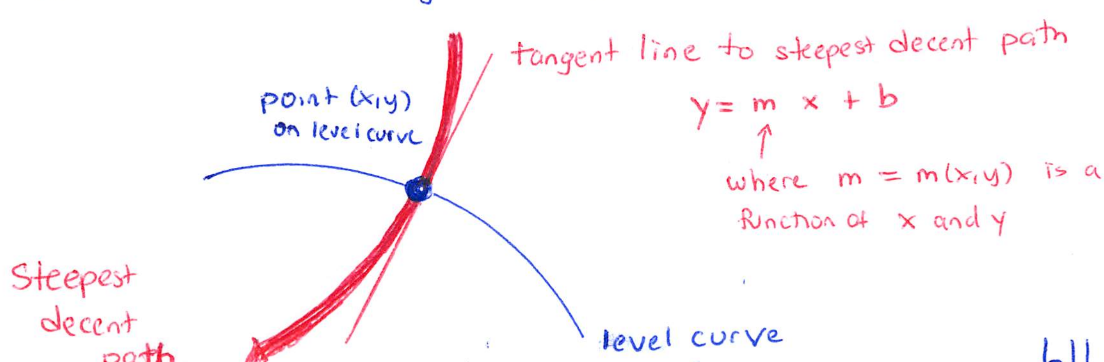
$$= -\langle 2x, 6y \rangle$$

$$= \langle -2x, -6y \rangle$$

We can use this direction to analyze the behavior of our path.



Then, at any point (x,y) on the steepest decent path, we can draw a tangent line, as seen below



We Know that the tangent line to this path has slope determined by the gradient at this point

$$m = \frac{-6y}{-2x} \begin{array}{l} \leftarrow \text{rise (change in y)} \\ \hline \leftarrow \text{run (change in x)} \end{array}$$

$$= \frac{3y}{x}$$

Then, the steepest decent path satisfies the Ordinary Differential equation

$$y'(x) = \frac{3y}{x}$$

$$\Rightarrow \frac{d}{dx}\left[y(x)\right] = \frac{3\,y(x)}{x}$$

with known initial point $y(3) = 4$. We see from the next page of these notes, the steepest descent path that satisfies this ODE is given by

$$y = \frac{4x^3}{27}$$

and ends at point $(0,0)$, the vertex of our paraboloid.

Solving ODE's via Separation of Variables

$$y' = \frac{3y}{x}$$

Preview of Coming Attractions

This technique is one of many solution methods discussed in Math 2B

$$\Rightarrow \quad \frac{y'}{3y} = \frac{1}{x}$$

$$\Rightarrow \quad \int \frac{1}{3y}\, dy = \int \frac{1}{x}\, dx$$

$$\Rightarrow \quad \frac{1}{3} \ln(y) = \ln(x) + c^{*}$$ for some constant $c^{*} \in \mathbb{R}$

$$\Rightarrow \quad \ln(y) = \ln(x^{3}) + c$$ for some $c \in \mathbb{R}$

$$\Rightarrow \quad y = A x^{3}$$ for some $A \in \mathbb{R}$

Since we know $y(3) = 4$, we can solve for A as follows

$$A \cdot [3]^{3} = 4 \Rightarrow A = \frac{4}{27} \Rightarrow \boxed{y(x) = \frac{4x^{3}}{27}}$$