Lesson 11: Directional Derivatives and the Gradient Handout
Reference: Brigg's "Calculus: Early Transcendentals, Second Edition"
Topics: Section 12.6: Directional Derivatives and the Gradient, p. 916 - 928

Let point $A(a, b, f(a, b))$ be a point on the surface $z = f(x, y)$, where $f(x, y)$ is a two-variable function. Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the $xy-$plane. In this lesson, we will learn how to define the derivative of f at the point $P_0(a, b)$ in the direction of our unit vector $\mathbf{u}$. In order to define our desired derivative, let's recall the formal definition of the ordinary derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This definition has two main components

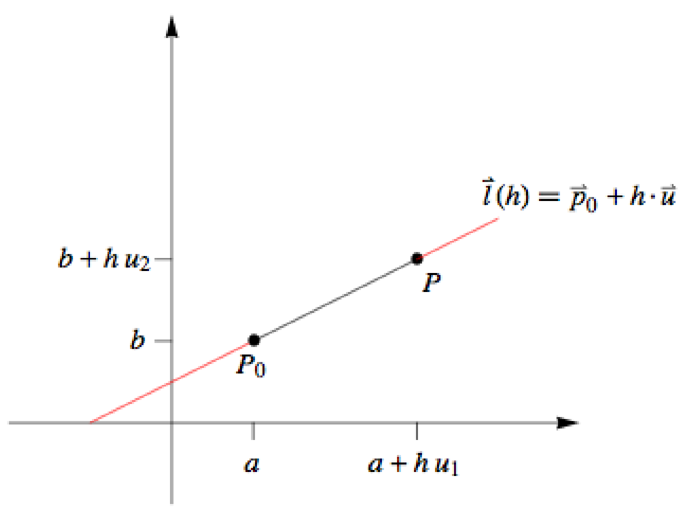
$$\frac{f(a+h) - f(a)}{h} \;:\; \text{Slope of secant line through points } A(a, f(a)) \text{ and } B(a+h, f(a+h))$$

$$\lim_{h \to 0} \;:\; \text{process that transforms the secant line into a tangent line}$$
$$\text{by forcing point } B \text{ to become point } A \text{ in the limit}$$

Let's apply this technology to the problem of finding a slope of a tangent line to a surface $z = f(x, y)$ at a point (a, b) in an arbitrary direction determined by the unit vector $\mathbf{u}$. We begin by constructing a secant line through two points on the surface. The first point results from evaluating f at $P_0(a, b)$. Our second point comes from traveling along the line $\ell(h)$ in $\mathbb{R}^2$ that passes through points P_0 and moves in the direction of vector $\mathbf{u}$. We recall from our discussion of lines in $\mathbb{R}^2$, we define the line

$$\ell(h) = \mathbf{p}_0 + h \cdot \mathbf{u} = \langle a, b \rangle + t \cdot \langle u_1, u_2 \rangle = \langle a + hu_1, b + hu_2 \rangle$$

Point P results from moving h units along line ℓ in the direction of $\mathbf{u}$. Thus, our second point is given by $P(a + hu_1, b + hu_2)$.



We can measure the distance between points $P_0(a, b)$ and $P(a + hu_1, b + hu_2)$ using the two norm, with

$$\left\| \overrightarrow{P_0P} \right\|_2^2 = (h\,u_1)^2 + (h\,u_2)^2 = h^2\,u_1^2 + h_2\,u_2^2 = h^2\ \left(u_1^2 + u_2^2\right) = h^2\ \|\mathbf{u}\|_2^2$$

However, since we assumed that $\mathbf{u}$ was a unit vector, we know that $\|\mathbf{u}\|_2 = 1$. Thus, the parameter h gives the distance from point P_0 to points P. Moreover, if $h > 0$, we know that point P comes from traveling in the same direction of $\mathbf{u}$ where $h < 0$ results in a P that comes from traveling $|h|$ units along $-\mathbf{u}$.

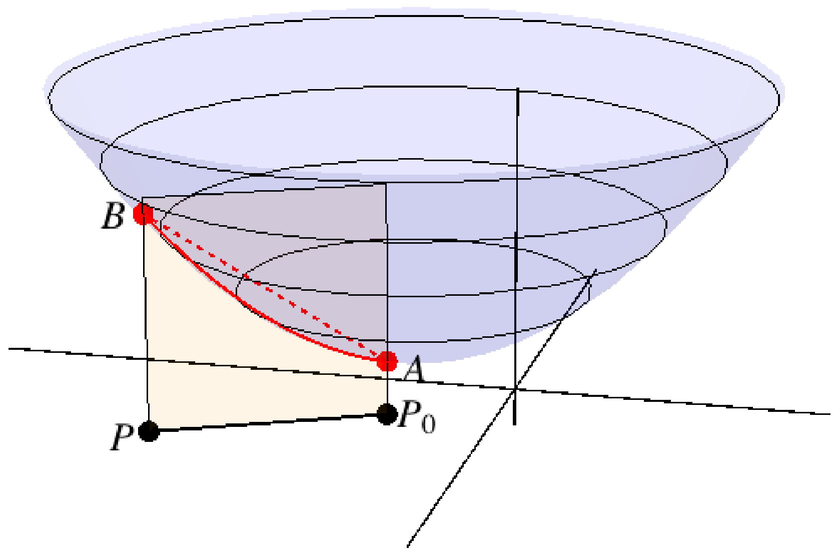
Now we can evaluate the function f at input point $P_0(a, b)$ to get the height of this point on the surface given by $f(a, b)$. This creates the point on the surface

$$A(a, b, f(a, b)).$$

We also find the value of function f at input point P which is given as $f(a + hu_1, b + hu_2)$ and yields a second point on our surface

$$B(a + hu_1, b + hu_2, f(a + hu_1, b + hu_2)).$$

Below, we visualize these two points on our surface.



Then, we can define a secant line through points A and B, which is shown as a dashed red line above. The slope of this secant line is

$$\frac{\text{rise}}{\text{run}} = \frac{\text{change in height between points } A \text{ and } B}{\text{distance between } P_0 \text{ and } P} = \frac{f(P) - f(P_0)}{\left\| P_0P \right\|_2}$$

We saw by our analysis above that this slope is given by the ratio

$$\frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

To transform the secant line into a tangent line, we force the points P toward the point P_0 by taking the limit as $h \to 0$.The slope of the tangent line can then be measured as

$$\lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}$$

This is the limit definition of the directional derivative that we wanted to create.

Definition. *p. 917* ***Directional Derivative***

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function that is differentiable at point (a, b). Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the $xy-$plane. The **directional derivative of f at (a, b) in the direction of u** is

$$D_{\mathbf{u}} f(a,b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}$$

provided this limit exists.

Now that we have a formal limit definition for the directional derivative, we might ask if it is possible to evaluate directional derivatives without explicitly taking a limit. The good news is that we can creatively use composite functions to express directional derivatives and, in doing so, apply the multivariable chain rule to express directional derivatives in terms of partial derivatives.

To this end, let us define a single variable function that results from evaluating $f(x, y)$ along the line $\ell(s) = \langle x(s), y(s) \rangle$. In other words, we will consider all points along the surface $z = f(x, y)$ where we constrain the input values of x and y to be on a line ℓ passing through point $P_0(a, b)$ in the direction of unit vector $\mathbf{u}$. We know that the input points on ℓ satisfy the parametric equations

$$\ell(s) = \langle x(s), y(s) \rangle$$

where each component of points on the line $\ell(s)$ are given by the parametric equations

$$x(s) = a + hu_1 \qquad \text{and} \qquad y(s) = b + hu_2$$

Then, we define the single-variable function $g(s)$ that results from taking a composite of $f(x, y)$ with $x(s)$ and $y(s)$, given by

$$g(s) = f(x, y) = f\Big(\, x(s)\, ,\, y(s) \,\Big)$$

Then, based on our work above, we can define the directional derivative as

$$\begin{aligned} g'(0) = \frac{d}{ds}\left[f\Big(\, x(s)\, ,\, y(s) \,\Big) \right]\Bigg|_{s=0} &= \left[\frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{yx}{ds} \right]\Bigg|_{s=0} \\ &= f_x(a,b)\, u_1 + f_y(a,b)\, u_2 \\ &= \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle \\ &= \nabla f(a,b) \cdot \mathbf{u} \end{aligned}$$

This observation leads to a much more efficient mechanism to evaluate directional derivatives.

Theorem 12.10. *p. 918* ***Directional Derivative***

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function that is differentiable at point (a, b). Let $\mathbf{u} = \langle u_1, u_2 \rangle$ be a unit vector in the $xy-$plane. The **directional derivative of f at (a, b) in the direction of u** is

$$D_{\mathbf{u}} f(a,b) = \langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle = \vec{\nabla} f(a,b) \cdot \mathbf{u}$$

Definition. *p. 919* ***Gradient (in Two Dimensions)*** *p. 919*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function, differentiable at point (x, y). The **gradient** of f at (x, y) is the vector-valued function

$$\vec{\nabla} f(x, y) = \langle\ f_x(x, y),\ f_y(x, y)\ \rangle = f_x(x, y)\,\mathbf{i} + f_y(x, y)\,\mathbf{j}$$

Theorem 12.11. *p. 920* ***The Gradient and Directions of Change***

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function that is differentiable at point (a, b), with $\nabla f(a, b) \neq \mathbf{0}$.

1. f has its maximum rate of increase at (a, b) in the direction of the gradient $\nabla f(a, b)$. The rate of change in this direction is $\|\nabla f(a, b)\|_2$

2. f has its maximum rate of decrease at (a, b) in the direction of $-\nabla f(a, b)$. The rate of change in this direction is $-\|\nabla f(a, b)\|_2$

3. The directional derivative is zero in any direction orthogonal to $\nabla f(a, b)$

Theorem 12.12. *p. 922* ***The Gradient and Level Curves***

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two-variable function that is differentiable at point (a, b). Then, the tangent line to the level curve of f at (a, b) is orthogonal to the gradient vector $\nabla f(a, b)$, provided that $\nabla f(a, b) \neq \mathbf{0}$.

Definition. *p. 924* ***Gradient (in Three Dimensions)*** *p. 919*

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a three-variable function, differentiable at point (x, y). The **gradient** of f at (x, y, z) is the vector-valued function

$$\vec{\nabla} f(x, y) = \langle\ f_x(x, y, z),\ f_y(x, y, z),\ f_z(x, y, z)\ \rangle$$

$$= f_x(x, y, z)\,\mathbf{i} + f_y(x, y, z)\,\mathbf{j} + f_z(x, y, z)\,\mathbf{k}$$