INA 4 The Ratio, Root and Comparison Test

Theoretic Problem

1. Ratio and Root Test

A. State the ratio test for convergence of a series with positive terms.

Solution:

For infinite series $\sum_{k=1}^{\infty} a_k$, with $a_k > 0$,

$$r = \lim_{k \to \infty} \frac{a_{k+1}}{a_k}$$

- if $0 \leq r < 1$, series converges;
- if $r > 1$, series diverges;
- if $r = 1$, inconclusive.

B. State the root test for convergence of a series with positive (nonnegative?) terms.

Solution:

For infinite series $\sum_{k=1}^{\infty} a_k$, with $a_k \geq 0$

$$\rho = \lim_{k \to \infty} \sqrt[k]{a_k}$$

- if $0 \leq \rho < 1$, series converges;
- if $\rho > 1$, series diverges;
- if $\rho = 1$, inconclusive.

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C. State the direct comparison test for convergence of a series with positive terms.

Solution: For infinite series $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$, $a_k > 0$, $b_k > 0$,

- if $0 < a_k \le b_k$, if $\sum_{k=1}^{\infty} b_k$ converges, $\sum_{k=1}^{\infty} a_k$ also converges;
- if $0 < b_k \le a_k$, if $\sum_{k=1}^{\infty} b_k$ diverges, $\sum_{k=1}^{\infty} a_k$ also diverges.

D. State the limit comparison test for convergence of a series with positive terms.

Solution: For infinite series $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$, $a_k > 0$, $b_k > 0$,

$$\lim_{k \to \infty} \frac{a_k}{b_k} = L$$

- if $0 < L < \infty$, $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ either both converges or both diverges.
- if $L = 0$, $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges
- if $L = \infty$, $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

Problem 2 – 12 Please see Jeff's Hand written Notes

Also question 1 (E) is on page 18 of Jeff's notes.

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Suggested Problems

13. Determine if the following series converges or diverges.

A. $\sum_{k=1}^{\infty} \frac{(k!)^3}{(3k)!}$ $\boxed{\text{diverges}}$.

Note the kth term includes $k!$, try Ratio Test: $(a_k > 0)$

$$r = \frac{a_{k+1}}{a_k} = \frac{[(k+1)!]^3}{[3(k+1)]!} \cdot \frac{(3k)!}{(k!)^3}$$

$$= \frac{[(k+1)!]^3 \cdot \cancel{(3k)!}}{3(k+1) \cdot \cancel{(3k)!} \cdot (k!)^3} = \frac{1}{3(k+1)} \cdot \left[\frac{(k+1)!}{k!}\right]^3$$

$$= \frac{1}{3} \cdot \frac{1}{k+1} \cdot (k+1)^3 = \frac{1}{3}(k+1)^2 > 1$$

By Ratio Test, this series diverges as $r > 1$.

B. $\sum_{k=1}^{\infty} \frac{1}{5^k - 3^k}$ $\boxed{\text{converges}}$

Note $5^k - 3^k > 2^k$, also $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges (geometric series, $r = \frac{1}{2}$)
(for any $k > 1$).

Try comparison Test: ($\frac{1}{5^k - 3^k} > 0$, $\frac{1}{2^k} > 0$)

Since $5^k - 3^k > 2^k$, $\frac{1}{5^k - 3^k} < \frac{1}{2^k}$

we know $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges,

Thus $\sum_{k=1}^{\infty} \frac{1}{5^k - 3^k}$ converges by Comparison Test.

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C. $\sum_{k=1}^{\infty} \frac{k^{100}}{(k+2)!}$ $\boxed{\text{Diverges}}$

$a_k = \frac{k^{100}}{(k+2)!}$ since k^{100} growth much faster than $(k+2)!$

when $\lim_{k\to\infty} a_k = \infty \neq 0$

By Divergence Test, this series Diverge.

D. $\sum_{k=2}^{\infty} \frac{1}{k^2 \ln k}$ $\boxed{\text{Converges}}$

Note $k^2 \ln k > k^2$ as $k \to \infty$,

so $\frac{1}{k^2 \ln k} < \frac{1}{k^2}$

we know $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (p-series, $p=2>1$)

By Comparison Test: $\left(\frac{1}{k^2} > 0, \frac{1}{k^2 \ln k} > 0\right)$

$\sum_{k=2}^{\infty} \frac{1}{k^2 \ln k}$ converges $\left(\text{as: } \frac{1}{k^2 \ln k} < \frac{1}{k^2} \text{ and } \sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges.}\right)$

E. $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}+1}$ $\boxed{\text{Converges}}$

Note $k^{3/2}+1 > k^{3/2}$,

so $\frac{1}{k^{3/2}+1} < \frac{1}{k^{3/2}}$

we know $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges (p-series, $p=\frac{3}{2}>1$)

By Comparison Test: $\left(\frac{1}{k^{3/2}} > 0, \frac{1}{k^{3/2}+1} > 0\right)$

$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}+1}$ converges.

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F. $\sum_{k=1}^{\infty} \frac{k}{e^k+3k}$ $\boxed{\text{converges}}$

Let $a_k = \frac{k}{e^k+3k}$, $b_k = \frac{k}{e^k}$

By Ratio Test, $r = \frac{k+1}{e^{k+1}} \cdot \frac{e^k}{k} = \frac{k+1}{e \cdot k} < 1$, $\sum_{k=1}^{\infty} \frac{k}{e^k}$ converges.

As $0 < a_k \leq b_k$,

By Comparison Test, $\sum_{k=1}^{\infty} \frac{k}{e^k+3k}$ converges.

G. $\sum_{k=1}^{\infty} 50k^{-k}$ $\boxed{\text{Converges}}$

Let $a_k = 50k^{-k} = \frac{50}{k^k} = 50 \cdot \frac{1}{k^k}$

Let $b_k = 50 \cdot \frac{1}{2^k}$ Notice $0 < a_k \leq b_k$

We know $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges (geometric, $r = \frac{1}{2}$)

so $\sum_{k=1}^{\infty} 50 \cdot \underbrace{\boxed{\frac{1}{2^k}}}_{b_k}$ converges $\Rightarrow$ $\sum_{k=1}^{\infty} 50 \cdot \underbrace{\boxed{\frac{1}{k^k}}}_{a_k}$ converges by Comparison Test.

H. $\sum_{k=1}^{\infty} \frac{2^k-1}{k^k+1}$ $\boxed{\text{Converges}}$

Let $a_k = \frac{2^k-1}{k^k+1}$, $b_k = \frac{2^k}{k^k}$, notice $0 < a_k \leq b_k$

we know $\sum_{k=1}^{\infty} \frac{2^k}{k^k}$ converges (geometric, $r = \frac{2}{k} < 1$)

so $\sum_{k=1}^{\infty} \frac{2^k-1}{k^k+1}$ converges by Comparison Test.

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