Name : ____________________ **Class Number:** _____

Math 1C: INA Lesson 3 Suggested Problems

Theoretic Problems: Discussed in-class

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1. **The divergence test**

   A. Create an analogy between the statements given below

   Statement 1A: If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$

   Statement 2A: If a person lives in California, then that person lives in the United States.

   What is the logical connection between these two statements?

   B. Please negate these two statements. For example, using statement 2A, what can we say if a person definitely does not live in the United States?

   C. Consider the two statements below:

   Statement 1B: If $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ converges.

   Statement 2B: If a person lives in the United States, then that person lives in California.

   Are statements 1B and 2B true? Are statements 1B and 2B equivalent to statements 1A and 2A, respectively? Explain your answers and provide example scenarios to give context to your ideas. Make sure that you have a strong grasp of the logical foundations that underpin the divergence theorem.

   D. Prove the tests for divergence (this is done for you in Jeff's Handwritten Notes for INA Lesson 3).

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2. **Derive the p-series test using the integral test** In this problem, we will determine for which real numbers $p \in R$ does the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

   converge or diverge? To do so, we will consider a few different scenarios for the value of $p \in \mathbb{R}$, discussed below.

   A. Suppose $p < 0$ is a negative number. Using the test for divergence, show that our series diverges.

   B. Suppose $p = 0$. Using the test for divergence, show that our series diverges.

   C. Suppose $0 < p \leq 1$. Use the integral test to show that the series diverges.

   D. Suppose $1 < p$. Use the integral test to show that the series converges.

   E. Use your work to state the results of the $p-$series test (Theorem 8.11 p. 632)

3. **Derive the remainder estimation technique associated with the integral test** In this problem, we will derive a method to estimate a series with positive terms that can be analyzed using the integral test. To this end, suppose that the single-variable function $f : [1, \infty) \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous, positive, and decreasing on it's domain $D = [1, \infty)$. Suppose we define a sequence $a_k = f(k)$ for all $k \in \mathbb{N}$ and that the associated convergent series converges to limit S with

$$\sum_{k=1}^{\infty} a_k = S.$$

Define the sequence of partial sums $\sum_{k=1}^{n} a_k = S_n$ and the remainder $R_n = S - S_n$.

A. Draw a diagram to represent a Riemann sum associated with the right-hand rule. Use this diagram to argue

$$R_n \leq \int_{n}^{\infty} f(x)dx$$

B. Draw a diagram to represent a Riemann sum associated with the left-hand rule. Use this diagram to argue

$$\int_{n+1}^{\infty} f(x)dx < R_n$$

C. Explain how the inequalities you discovered in parts A and B above give rise to the integral test for infinite series.

D. Explain how you can use the inequality you found in part A to approximate the value of a convergent series that can be analyzed using the integral test. How is this result related to the $p-$series test?

Problems Solved in Jeff's Handwritten Notes

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4. Example 8.4.1a p. 628: Use the test for divergence to show that the series below diverges:

$$\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}$$

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5. Example 8.4.1c p. 826: Consider the harmonic series:

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

Define the sequence of partial sums $S_n = \sum_{k=1}^{n} \frac{1}{k}$

A. Following the argument discussed in class, show that

$$S_2 \geq 1 + \frac{1}{2}, \quad S_4 \geq 1 + \frac{2}{2}, \quad S_8 \geq 1 + \frac{3}{2}, \quad S_{16} \geq 1 + \frac{4}{2}, \quad S_{32} \geq 1 + \frac{5}{2}, \quad S_{2^m} \geq 1 + \frac{m}{2}$$

B. Use your work in part A to show that the harmonic series diverges.

C. Explain why the harmonic series shows that **Statement 1B** from problem 1 above is false. Explain why this doesn't contradict **Statement 1A**.

D. Use the integral test to show that the harmonic series diverges.

E. Use the $p-$series test to show that the harmonic series diverges.

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6. **The Basel Problem-** Exercise 8.4.66 p. 639: Use the integral test to show that the infinite series below converges.

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

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7. Example 8.4.11 p. 638: Determine if the following series converges:

$$\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$$

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8. Example 8.4.3 p. 632: Use the $p-$series test to determine if the following series converge?

A. $\sum_{k=1}^{\infty} k^{-3}$

B. $\sum_{k=1}^{\infty} \frac{1}{\sqrt[4]{k^3}}$

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9. Example 8.4.4 p. 635: How many terms of the convergent $p-$series

$$\sum_{k=1}^{\infty} \frac{1}{k^3}$$

must be summed to get within $\epsilon = 10^{-3}$ of the exact value of this series.

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10. Example 8.4.5 p. 637: Find the sum of the series:

$$\sum_{k=1}^{\infty} \left[\frac{3}{n^2+n} + \frac{1}{2^n} \right]$$

must be summed to get within $\epsilon = 10^{-3}$ of the exact value of this series.

Suggested Problems

9. How many terms of the convergent $p-$series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

must be summed to get within $\epsilon = \frac{5}{10^4}$ of the exact value of this series. Remember, this series sums to the exact value $\frac{\pi^2}{6}$. The exact value comes from a study of the Riemann Zeta function (for a fun adventure, please google search the phrase "Riemann Zeta function" and enjoy the results).

10. Exercise 8.4.31 p. 638

11. Exercise 8.4.13 p. 638

12. Exercise 8.4.21 p. 638

13. Exercise 8.4.35 p. 638

14. Exercise 8.4.47 p. 638

Optional Challenge Problems

8. Exercise 8.4.65 p. 639
9. Exercise 8.4.66 p. 639
10. Exercise 8.4.67 p. 639
11. Exercise 8.4.72 p. 639