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total: 4h 30min.

INA 3 Divergence & Integral Test

1. The divergence test

A. Create an analogy between the statements given below

statement 1A: If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$

statement 2A: If a person lives in California, then that person lives in the United states.

what is the logical connection between these two statements?

Solution:

Both statements 1A & 2A has the form:

If P then q (written as $P \to q$).

B. Please write contrapositive statement of these two statements.

For example, what can we say if a person definitely does not live in the United States.

Solution

Recall contrapositive: if P, then q

the contrapositive statement is: if not q, then not P.

So the contrapositive for 1A:

If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=0}^{\infty} a_k$ diverges.

Contrapositive for 2A:

If the person does not live in US, then he not live in the California.

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C. Consider the two statements below:

statement 1B: If $\lim_{k\to\infty} a_k = 0$, then $\sum_{k=1}^{\infty} a_k$ converges.

statement 2B: If a person lives in US, then that person lives in CA.

Are statement 1B and 2B true? Are statement 1B and 2B equivalent to Statement 1A and 2A? Explain and provide example scenarious.

$\boxed{\text{Solution}}$

The statement 1B and 2B are not true, and not equivalent to 1A and 1B

Explain & example:

"Harmonic series"

- statement 1B is false for $\sum_{k=1}^{\infty} \frac{1}{k}$

  the limit of $a_k \to 0$, but series diverge;

- statement 2B is false if the person lives in US, but live in New York (Not in California). :)

Note:

The terms "convers", "inverse", "contrapositive"

original: if p then q

Converse: if q then p

inverse: if not p then not q

Contrapositive: if not q then not p

- original & contrapositive equivalent; converse & inverse equivalent.

※ equivalent: either both true or both false

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D. Prove the test for divergence

solution

Recall: Test for divergence

If the infinite series Σa_k converge, then $\lim_{k\to\infty} a_k = 0$ (I)

Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then infinite series Σa_k diverge. (II)

Note: statement (I) and (II) are contrapositive statement, so we can prove (I) and know (II) also true.

Prove (I):

If Σa_k converge, we have $\sum_{k=1}^{\infty} a_k = S$ exists $(S \in \mathbb{R})$

$$a_n = S_n - S_{n-1}$$

$$= (a_1 + a_2 + a_3 + \cdots + a_n) - (a_1 + a_2 + \cdots + a_{n-1})$$

$$\sum_{k=1}^{\infty} a_k = S \Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = S$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1})$$

$$= \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S$$

$$= 0$$

Note: "limit property"

$\lim_{x\to a} (f(x) + g(x))$

$= \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

So we proved if Σa_k converge, $\lim_{k\to\infty} a_k = 0$.

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2. Derive the p-series test using integral test.

Determine for which real number $p \in \mathbb{R}$ does the series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converge or diverge? Consider scenarios:

A. Suppose $p<0$ (negative number). Use test for divergence, show the series diverges.

$\boxed{\text{solution}}$

$$\lim_{n\to\infty} a_n = \lim_{k\to\infty} \frac{1}{k^p} = \lim_{k\to\infty} k^{-p}$$

Let $r = -p$, $r>0$

$$\lim_{n\to\infty} a_n = \lim_{k\to\infty} k^r = \infty$$

so $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges by divergence test.

B. Suppose $p=0$. use test for divergence, show series diverges.

$\boxed{\text{solution}}$

$$\lim_{n\to\infty} a_n = \lim_{k\to\infty} \frac{1}{k^p} = \lim_{k\to\infty} k^{-p} = \lim_{k\to\infty} k^0 = \lim_{k\to\infty} 1 = 1$$

since $\lim_{n\to\infty} a_n \neq 0$, $\sum a_n$ diverges.

so $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges by divergence test.

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continue #2

C. Suppose $0 < p \le 1$. Use the integral test to show series diverges.

$\boxed{\text{Solution}}$

Recall integral test: $\sum_{k=1}^{\infty} a_k$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

Given $f(x)$ continuous, positive, decreasing. $(a_k = f(k))$

First test conditions:

$f(x) = \frac{1}{x^p} = x^{-p}$

☑ continuous for $x \geqslant 1$
☑ positive
☑ decreasing (as x increase, x^p increase, $\frac{1}{x^p}$ decrease)

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} x^{-p}\,dx = \frac{1}{1-p} x^{1-p} \Big|_1^{\infty}$$

since $0 < p \le 1$, $-1 \le -p < 0$, $0 \le 1-p < 1$

$$\int_1^{\infty} f(x)\,dx = \frac{1}{1-p} x^{1-p} \Big|_1^{\infty}$$

$$= \frac{1}{1-p}\left[\infty^{1-p} - 1^{1-p} \right]$$

$$= \infty$$

so we know $\sum_{k=1}^{\infty} a_k$ also diverge by integral test.

D. Suppose $p > 1$. use integral test to show series converge.

$\boxed{\text{Solution}}$ Like discussed above, we have:

$$\int_1^{\infty} f(x)\,dx = \frac{1}{1-p} x^{1-p} \Big|_1^{\infty}$$

since $p > 1$, $1-p < 0$, $(p-1 > 0)$

$$\text{so } \frac{1}{1-p} x^{1-p} = \frac{1}{1-p} \cdot \frac{1}{x^{p-1}} = -\frac{1}{(p-1)\cdot x^{p-1}}$$

$$\int_1^{\infty} f(x)\,dx = -\frac{1}{(p-1)\cdot \infty^{p-1}} - \left(-\frac{1}{(p-1)\cdot 1^{p-1}} \right)$$

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Continue #2 (D)

$$\int_1^{\infty} f(x)\,dx = -\frac{1}{(P-1)\cdot\infty^{P-1}} - \left(-\frac{1}{(P-1)\cdot 1^{P-1}}\right)$$

$$= \frac{-1}{(P-1)\cdot\infty^{P-1}} + \frac{1}{P-1}$$

Note: $\infty^{P-1} \to \infty$ since $P-1>0$.

$$= \frac{-1}{\infty} + \frac{1}{P-1}$$

$$= 0 + \frac{1}{P-1}$$

$$= \frac{1}{P-1}$$

So we know Σa_k also converge by integral test.

E. Use your work to state the result of P-series test.

$\boxed{\text{Solution}}$

$\sum_{k=1}^{\infty} \frac{1}{k^P}$ converges when $P>1$ and diverges when $P \leq 1$.

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3. Derive the remainder estimation technique associated with integral test. Suppose single variable function $f: [1, \infty) \subseteq \mathbb{R} \to \mathbb{R}$ is continous, positive and decreasing on its domain $D: [1, \infty)$

Define sequence $a_k = f(k)$ for $k \in \mathbb{N}$, the associated convergent series converges to limit S with $\sum_{k=1}^{\infty} a_k = S$

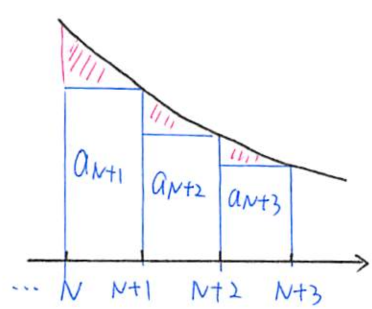
Define the sequence of partial sums $\sum_{k=1}^{\infty} a_k = S_n$, remainder $R_n = S - S_n$

A. Draw a diagram to represent a Riemann Sum associated with Right-Hand Rule. Use Diagram to argue $R_n \leq \int_n^{\infty} f(x)\,dx$

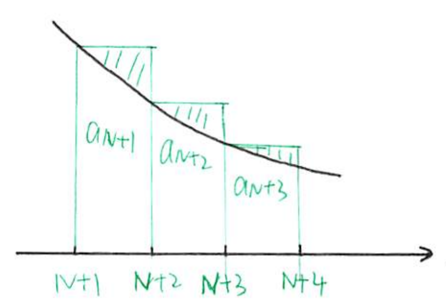
B. Draw a diagram to represent a Riemann Sum associated with Left-Hand Rule. Use Diagram to argue $\int_{n+1}^{\infty} f(x)\,dx < R_n$.

$\boxed{\text{Solution for A \& B}}$

(A).



(B)



C. Explain how the inequalities you discovered in Part A & B give rise to integral test for infinite series.

$\boxed{\text{Solution}}$

From Part A & B we have

Note: $R_n \geq 0$

$$\begin{cases} R_n \leq \int_n^{\infty} f(x)\,dx & \text{As } \int_1^{\infty} f(x)dx \text{ converge, } \int_n^{\infty} f(x)dx = 0, \quad R_n = 0, \quad \sum_{k=1}^{\infty} a_k \text{ converge} \\ \int_1^{\infty} f(x)\,dx < R_n & \text{As } \int_1^{\infty} f(x)dx \text{ diverge, } \int_{n+1}^{\infty} f(x)\,dx = \infty, \quad R_n = \infty, \quad \sum_{k=1}^{\infty} a_k \text{ diverge.} \end{cases}$$

The convergence behavior depends on tails.

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~~so~~ $\cancel{\sum_{k=1}^{\infty} a_k}$ ~~is bound above & below~~;

~~As~~ $\cancel{n\to\infty}$, $\cancel{\lim_{n\to\infty}\left[S_n + \int_{n+1}^{\infty} f(x)\,dx\right] = \lim_{n\to\infty}\left[S_n + \int_{n}^{\infty} f(x)\,dx\right] = S}$

~~So~~ $\cancel{\lim_{k\to\infty} \sum_{k=1}^{\infty} a_k = S}$ ~~by squeeze theorem~~.

$\Rightarrow$ $\cancel{\sum_{k=1}^{n} a_k}$ ~~and~~ $\cancel{\int_1^{\infty} f(x)\,dx}$ ~~either converge or diverge at same time~~.

D. Explain how you can use the inequality you found in part A to approximate the value of a convergent series that can be analyzed using integral test. How is the result related to p-series test?

$\boxed{\text{Solution}}$

From part A we have $R_n \le \int_n^{\infty} f(x)\,dx$

$$S_n + R_n \le S_n + \int_n^{\infty} f(x)\,dx \le \int_1^{\infty} f(x)\,dx$$

$$\sum_{k=1}^{\infty} a_k \le \int_1^{\infty} f(x)\,dx$$

So we can estimate the value of convergent series using integral. when $f(x)$ is continuous, positive & decreasing.

when $P > 1$, $\int_1^{\infty} \frac{1}{k^P}$ converges, $\sum_{k=1}^{\infty} a_k$ must also converge

since $\sum_{k=1}^{\infty} a_k \le \int_1^{\infty} f(x)\,dx$ ($\int_1^{\infty} f(x)\,dx$ sets as upper bound).

Question 4-8

please see Jeff's Hand Written Notes.

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9. How many terms of the convergent p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ must be summed to get within $\varepsilon = \frac{5}{10^4}$ of the exact value of this series. The series sums to exact value $\frac{\pi^2}{6}$

$\boxed{\text{Solution}}$ First note $f(k)$ continuous, positive & decreasing, so integral test applies.

Recall the remaining term $R_n \le \int_n^{\infty} f(x)\,dx$

Let $\int_n^{\infty} f(x)\,dx < \frac{5}{10^4}$

$$\int_n^{\infty} \frac{1}{x^2}\,dx < \frac{5}{10^4}$$

$$-\frac{1}{x}\Big|_n^{\infty} < \frac{5}{10^4}$$

$$-\frac{1}{\infty} - \left(-\frac{1}{n}\right) < \frac{5}{10^4}$$

$$\frac{1}{n} < \frac{5}{10^4}$$

$$n > \frac{10^4}{5} = 2000$$

So we need at least 2001 terms to get within $\varepsilon = \frac{5}{10^4}$ of exact value.

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