INA2 Infinite Series

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Total: 103 min.

1. Geometric series test

Consider the infinite series $\sum_{k=1}^{\infty} a \cdot r^{k-1}$

A. Derive the geometric sum formula given by

$$S_n = \sum_{k=1}^{n} a \cdot r^{k-1} = a \cdot \frac{1-r^n}{1-r}$$

$\boxed{\text{Solution}}$

Note: we want to cancel the terms

$$S_n = a + \cancel{a} \cdot r + \cancel{a} \cdot r^2 + \cdots + \cancel{a} \cdot r^{n-1}$$

$$r \cdot S_n = \cancel{a} \cdot r + \cancel{a} \cdot r^2 + \cdots + \cancel{a} \cdot r^{n-1} + a \cdot r^n$$

$$S_n - r \cdot S_n = a - a \cdot r^n$$

$$S_n(1-r) = a - a \cdot r^n$$

$$S_n = a \frac{1-r^n}{1-r}$$

B. Use part A and the limit of a geometric sequence to derive the geometric series test.

$\boxed{\text{Solution}}$

Case I. when $r=1$ $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a \cdot r^{n-1} = \sum_{k=1}^{\infty} a = a \cdot n$

when $n \to \infty$, $a \cdot n \to \infty$, series divergent.

Case II. when $-1<r<1$, $r^n \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} \left(a \frac{1-r^n}{1-r} \right) = a \cdot \frac{1}{1-r}$$

Case III. when $r \leq -1$ or $r>1$

$|r|>1$, the geometric series diverge.

$\Rightarrow$ Conclusion: if $|r| \geq 1$, series diverge; if $|r|<1$, series converge.

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23 min.

Question 2 - 8

please see Jeff's Handwritten notes.

Note: #8

on Jeff's notes, k starts with 0

$$\sum_{k=0}^{\infty} \frac{1}{16k^2+8k-3} = -\frac{1}{4}$$

If we start at $k=1$, the limit

$$\sum_{k=1}^{\infty} \frac{1}{16k^2+8k-3} = \frac{1}{12} - \frac{1}{4(4k-3)} = \frac{1}{12}$$

The starting point will not influence convergence / divergence,

but the limit will be different.

9. Write the repeating decimal $1.00\overline{39} = 1.00393939\cdots$ as a geometric series. Then use geometric series test to write this repeating decimals as a fraction.

$\boxed{\text{solution}}$

$$1.00\overline{39} = 1 + 0.0039 + 0.000039 + \cdots$$

$$= 1 + \frac{39}{10000} + \frac{39}{1000000} + \cdots$$

$$= 1 + \frac{39}{10^4} + \frac{39}{10^6} + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \frac{39}{10^{2n+2}}$$

$$\frac{1}{10^{2n+2}} = \frac{1}{10^{2n}\cdot 10^2} = \frac{1}{(10^2)^n \cdot 10^2} = \frac{1}{100^n \cdot 100} = \frac{1}{100}\cdot\left(\frac{1}{100}\right)^n$$

$$\frac{39}{10^{2n+2}} = \frac{39}{100^n \cdot 100} = \frac{39}{100}\cdot\left(\frac{1}{100}\right)^n = \frac{39}{10^4}\cdot\left(\frac{1}{100}\right)^{n-1}$$

$$a = \frac{39}{10^4},\quad r = \frac{1}{100},\quad S_n = \frac{39}{10^4}\cdot\frac{1-\left(\frac{1}{100}\right)^n}{1-\frac{1}{100}} = \frac{39}{9900}$$

so $1.00\overline{39} = 1 + \frac{39}{9900} = \boxed{\frac{9939}{9900}}$.

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10. Find the limit of series: $\sum_{k=3}^{\infty} \frac{1}{9k^2+15k+4}$

$\boxed{\text{Solution}}$

$$\frac{1}{9k^2+15k+4} = \frac{1}{(3k+1)(3k+4)} = \frac{A}{3k+1} + \frac{B}{3k+4}$$

$$A(3k+4) + B(3k+1) = 1$$

$$\Rightarrow \begin{cases} 3k(A+B) = 0 \\ 4A + B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{3} \\ B = -\frac{1}{3} \end{cases}$$

partial fraction decomposition.

So we have:

$$\sum_{k=3}^{\infty} \frac{1}{9k^2+15k+4} = \sum_{k=3}^{\infty} \frac{\frac{1}{3}}{3k+1} - \frac{\frac{1}{3}}{3k+4} = \sum_{k=3}^{\infty} \frac{1}{3(3k+1)} - \frac{1}{3(3k+4)}$$

$$= \frac{1}{3(3\cdot3+1)} - \cancel{\frac{1}{3(3\cdot3+4)}} + \cancel{\frac{1}{3(3\cdot4+1)}} - \cancel{\frac{1}{3(3\cdot4+4)}} + \cdots$$

$$+ \cancel{\frac{1}{3(3k+1)}} - \frac{1}{3(3k+4)}$$

Note: they are same

$$= \frac{1}{30} - \frac{1}{9k+12}$$

As $k \to \infty$, $9k+12 \to \infty$

$\frac{1}{9k+12} \to 0$.

$$= \boxed{\frac{1}{30}}$$

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March 7 11:18AM – 11:28AM

10 min

11. Find the limit of series: $\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \cdot 5^{3-k}$

$\boxed{\text{solution}}$

$$\left(\frac{1}{4}\right)^k \cdot 5^{3-k} = \frac{1}{4^k} \cdot \frac{5^3}{5^k} = 5^3 \cdot \frac{1}{4^k \cdot 5^k}$$

$$= 5^3 \cdot \frac{1}{(4 \cdot 5)^k}$$

$$= 125 \cdot \left(\frac{1}{20}\right)^k$$

Recall:

$\left(\frac{1}{a}\right)^n = \frac{1^n}{a^n} = \frac{1}{a^n}$

$a^n \cdot b^n = (a \cdot b)^n$

$$\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \cdot 5^{3-k} = \sum_{k=0}^{\infty} 125 \cdot \left(\frac{1}{20}\right)^k = \sum_{k=1}^{\infty} 125 \cdot \left(\frac{1}{20}\right)^{k-1}$$

This is a geometric series with $a = 125$, $r = \frac{1}{20}$.

$$S_n = a \cdot \frac{1 - r^n}{1 - r} = 125 \cdot \frac{1 - \frac{1}{20^n}}{1 - \frac{1}{20}}$$

$$= 125 \cdot \frac{1}{\frac{19}{20}}$$

$$= 125 \cdot \frac{20}{19}$$

$$= \boxed{\frac{2500}{19}}$$

As $n \to \infty$

$20^n \to \infty$

$\frac{1}{20^n} \to 0$.

IMA2 Infinite Series

March 7 11:30 AM - 11:40 AM

10 min