I NA lesson 1 Sequences

1. Define the two major problems we work on in sequences & series.

A. Please describe the two types of problems we will study in "Introduction to Numerical Analysis" part of this class.

$\boxed{\text{Solution}}$

I. The numerical evaluation problem.
Given function $f(x)$, calculate a numerical approximation $f(c)$ in a certain error range.

II. Ordinary differential problem.

Given ODE, find solution of the equation.
~~How: Represent function using power series.~~

B. Please recall the definition of a sequence. Explain how sequences are related to the problems you discussed above.

$\boxed{\text{solution}}$

A sequence $\{a_n\}_{n=1}^{\infty}$ is a function $a : \mathbb{N} \to \mathbb{R}$

Sequence is an ordered list of numbers.

To solve the two questions above, we represent the function using (power) series.

C. Please recall the three different representations of a sequence.

$\boxed{\text{Solution}}$

I, ordered set $\{a_1, a_2 \cdots a_n\}$

II. recurrence relation $a_{n+1} = f(a_n)$ * write a_{n+1} as function of a_n, a_1 must be given.

III. explicit formula. $a_n = f(n)$. * write a_n as function of n.

Question 2-8 is solved in Jeff's Handwritten Notes.

Note: #6 instead of theorem 8.2, use theorem 8.1 ?

9. Define the sequence $\{a_n\}_{n=1}^{\infty}$ using the explicit formula

$$a_n = 10 + \frac{1}{n}$$

A. show that this sequence is decreasing.

$\boxed{\text{Solution}}$

To show decreasing we need to show $a_{n+1} < a_n$.

$$a_{n+1} = 10 + \frac{1}{n+1}$$

$$\begin{aligned} a_{n+1} - a_n &= \left(10 + \frac{1}{n+1}\right) - \left(10 + \frac{1}{n}\right) \\ &= \frac{1}{n+1} - \frac{1}{n} \\ &= \frac{n-(n+1)}{n(n+1)} \\ &= \frac{-1}{n(n+1)} \end{aligned}$$

since $n > 0$, $n(n+1) > 0$

$$\Rightarrow \frac{-1}{n(n+1)} < 0$$

so we have: $a_{n+1} - a_n < 0$.

$$\Rightarrow a_{n+1} < a_n \quad \square$$

B. Show that this sequence is bounded

$\boxed{\text{Solution}}$

To show bounded we need to show $|a_n| \leq M$

since $\frac{1}{n} \leq 1$, $10 + \frac{1}{n} \leq 11$ $\square$

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12 min

Continue #9

C. show that this sequence is nonincreasing.

$\boxed{\text{solution}}$

Note if a sequence is decreasing, it must be nonincreasing.

(which is already proved in part A).

Another way to prove:

By definition of nonincreasing, $a_{n+1} \le a_n$

$$a_{n+1} = 10 + \frac{1}{n+1}$$

$$a_n = 10 + \frac{1}{n}$$

since $\frac{1}{n+1} \le \frac{1}{n}$

$$10 + \frac{1}{n+1} \le 10 + \frac{1}{n}$$

$$\Rightarrow \quad a_{n+1} \le a_n \qquad \square$$

D. show that this sequence is monotonic.

$\boxed{\text{solution}}$

By definition, $\{a_n\}_{n=1}^{\infty}$ is monotonic if it is either nonincreasing or nondecreasing.

From Part C we know it is nonincreasing.

Thus the sequence is monotonic.

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8min -11:11AM.

10. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ defined by the recurrence relation $a_1 = 3$, $a_{n+1} = 0.5\, a_n$

(i) Find the first 5 terms of this sequence

solution

$a_1 = 3$

$a_2 = 0.5 \times a_1 = \frac{3}{2}$

$a_3 = 0.5 \times a_2 = \frac{3}{4}$

$a_4 = 0.5 \times a_3 = \frac{3}{8}$

$a_5 = 0.5 \times a_4 = \frac{3}{16}$

(ii) Find an explicit formula for a_n in terms of n.

solution

Recall explicit form : $a_n = f(n)$

$a_1 = 3 = \frac{3}{2^0}$

$a_2 = \frac{3}{2} = \frac{3}{2^1}$

$a_3 = \frac{3}{4} = \frac{3}{2^2}$

$a_4 = \frac{3}{8} = \frac{3}{2^3}$

$\Rightarrow \boxed{a_n = \frac{3}{2^{n-1}}}$

(iii) Find the $\lim_{n\to\infty} a_n$

solution

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3}{2^{n-1}} = 0$$

when $n \to \infty$, $2^{n-1} \to \infty$, $\frac{3}{\infty}$ leads to 0.

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8 min

11. Suppose we have sequence $\{a_n\}_{n=1}^{\infty}$ defined by the ordered set

$$\left\{5, \frac{5}{2}, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}, \ldots\right\}.$$

A. Find the next two terms of this sequence.

$\boxed{\text{solution}}$

Notice $a_2 = a_1 \cdot \frac{1}{2}$,

$a_3 = a_2 \cdot \frac{1}{2}$

So the next two terms are: $\frac{5}{16} \times \frac{1}{2} = \boxed{\frac{5}{32}}$; $\frac{5}{32} \times \frac{1}{2} = \boxed{\frac{5}{64}}$.

B. Represent this sequence as a one-term recurrence relation.

$\boxed{\text{solution}}$ $a_{n+1} = \frac{1}{2} a_n$

C. Find an explicit formula for a_n in terms of n.

$\boxed{\text{Solution}}$

Notice
$$a_1 = 5 = \frac{5}{2^0}$$
$$a_2 = \frac{5}{2} = \frac{5}{2^1} \Rightarrow \boxed{a_n = \frac{5}{2^{n-1}}}$$
$$a_3 = \frac{5}{4} = \frac{5}{2^2}$$
$$\ldots$$

D. Find the $\lim\limits_{n\to\infty} a_n$

$\boxed{\text{Solution}}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5}{2^{n-1}} = 0$$

Because as $n\to\infty$, $2^{n-1} \to \infty$, so $\frac{5}{\infty} \to 0$.

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5 min

12. Suppose we have a sequence $\{a_n\}_{n=1}^{\infty}$ defined by the recurrence relation

$a_1 = 7, \qquad a_{n+1} = -1 \cdot a_n$

A. Represent this sequence as an ordered set (first 6 terms)

$\boxed{\text{Solution}}$

$a_1 = 7$

$a_2 = -1 \cdot a_1 = -7$

$a_3 = -1 \cdot a_2 = 7$

$a_4 = -1 \cdot a_3 = -7$

$a_5 = -1 \cdot a_4 = 7$

$a_6 = -1 \cdot a_5 = -7$

so the sequence is:

$\{7, -7, 7, -7, 7, -7 \cdots\}$.

B. Find an explicit formula for a_n in terms of the index n.

$\boxed{\text{Solution}}$ $a_n = (-1)^{n+1} \cdot 7$ or $(-1)^{n-1} \cdot 7$

Because for even term, the value is positive.

C. Find the $\lim\limits_{n \to \infty} a_n$

$\boxed{\text{Solution}}$

By Theorem 8.3 limit of a geometric sequence,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & -1 < r < 1 \\ 1 & r = 1 \\ \text{DNE} & r \leq -1 \text{ or } r > 1 \end{cases}$$

Here we have $a_n = 7 \cdot (-1)^{n+1} = 7 \cdot (-1) \cdot (-1)^n = -7 \cdot (-1)^n$

$(-1)^n$ is a geometric sequence,

$r = -1$, $\lim\limits_{n\to\infty} (-1)^n$ DNE

Thus the $\lim\limits_{n\to\infty} a_n$ DNE.

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10min

13. (12 in the assignment)

Evaluate the limit or state that the limit DNE for sequence

$$a_n = \int_1^n x^{-2}\,dx$$

Solution

$$a_n = \int_1^n x^{-2}\,dx = -x^{-1}\Big|_1^n = -\frac{1}{x}\Big|_1^n$$

$$= -\frac{1}{n} - (-1)$$

$$= 1 - \frac{1}{n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 - \frac{1}{n} = \boxed{1}$$

Because as $n \to \infty$, $\frac{1}{n} \to 0$

Thus the limit of this sequence is 1.

March 4 11:36–11:40AM
4 min