True/False

For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

1. **(T)** F $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$

2. T **(F)** If $\sum_{n=0}^{\infty} a_n$ is divergent, then $\sum_{n=0}^{\infty} |a_n|$ is divergent.

3. T **(F)** If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

4. **(T)** F If $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

5. T **(F)** If $0 \le a_n \le b_n$ for all n and $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

6. **(T)** F If $a_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then $\lim_{n\to\infty} a_n = 0$.

7. T **(F)** The ratio test can be used to determine if the series $\sum_{n=0}^{\infty} \frac{1}{n^3}$ converges

8. T **(F)** The series $\sum_{n=1}^{\infty} 3ne^{-n^2}$ diverges

Multiple Choice

For the problems below, circle the correct response for each question. After you've chosen your answer, mark your answer on your Scantron form.

9. For which of the following values of k do both $\sum_{n=0}^{\infty}\left(\frac{3}{k}\right)^n$ and $\sum_{n=0}^{\infty}\frac{(3-k)^n}{\sqrt{n+3}}$ converge?

A. None   B. 2   C. 3   **D. 4**   E. 5

10. The series $\sum_{n=1}^{\infty}\frac{1}{n^\alpha}$ converges if and only if

A. $1 < \alpha$ B. $\alpha < 1$ C. $-1 < \alpha < 1$ D. $\alpha \geq 1$ E. $-1 < \alpha$

11. Which of the following series converge?

I. $\sum_{n=2}^{\infty}\frac{1}{n\left(\ln(n)\right)^4}$ II. $\sum_{n=1}^{\infty}\frac{3+\sin(n)}{n^4}$ III. $\sum_{n=1}^{\infty}\frac{7n^2-5}{e^n(n+3)^2}$

A. I only   B. I and II only   C. II and III only   D. I and III only   **E. I, II, and III**

12. Which of the following series converge?

I. $\sum_{n=2}^{\infty}\frac{1}{n^2\cdot\ln(n)}$ II. $\sum_{n=2}^{\infty}\frac{1}{n\cdot\ln(n)}$ III. $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}\cdot\ln(n)}$

A. I only   B. II only   C. I and II only   D. I and III only   E. I, II, and III

13. Which of the following is the radius of convergence for the series $\sum_{n=0}^{\infty}\frac{(-1)^n n!\,x^n}{n^n}$?

A. 0   B. $\frac{1}{e}$ C. 1   **D.** e E. ∞

14. What are the values of x for which the series

$$(x+1) - \frac{(x+1)^2}{2} + \frac{(x+1)^3}{3} - \cdots + \frac{(-1)^{n+1}(x+1)^n}{n} + \cdots$$

converges?

A. $-2 < x \leq 0$ B. All real numbers C. $-2 < x < 0$ D. $-2 \leq x < 0$ E. $-2 \leq x \leq 0$

---

15. For what real values of k could the series

$$\sum_{n=1}^{\infty} \frac{\ln(n) + n^3}{n^{5k} + 4}$$

converges?

A. $k \geq 1$ B. $k < 1$ C. all real k D. $k > 1/5$ **E.** $k > 4/5$

---

16. Find the values of x for which the series $\sum_{n=1}^{\infty} (x-1)^n$:

A. $-2 < x < 0$ B. $0 < x \leq 2$ **C.** $0 < x < 2$ D. $0 \leq x \leq 2$ E. $-2 \leq x < 0$

---

17. What are all values of x for which the series

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{\sqrt[3]{n^2}}$$

converge?

A. $-1 \leq x \leq 1$ B. $0 < x < 1$ **C.** $0 \leq x < 1$ D. $0 < x \leq 1$ E. $-1 \leq x \leq 1$

---

18. The series $\sum_{n=0}^{\infty} r^n$ converges if and only if:

A. $-1 < r < 1$ B. $-1 \leq r \leq 1$ C. $-1 \leq r < 1$ D. $-1 < r \leq 1$ E. $r < 1$

19. Consider the series

$$\frac{1}{2}+\frac{(x-25)}{4}+\frac{(x-25)^2}{8}+\cdots+\frac{(x-25)^n}{2^{n+1}}+\cdots$$

Which of the following statements are true about this series?

I. The interval where the series converges is $23 < x < 27$

II. The sum of the series where it converges is equal to $\dfrac{1}{27-x}$

III. If $x = 28$, the series converges to $\dfrac{1}{3}$

IV. The interval where the series converges is $21 < x < 29$

V. The series diverges

A. I only B. III only **C. I and II only** D. I and IV only E. V only

---

20. How many terms of the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}n^{-2}$$

must we add in order to be sure that the partial sum s_n is within 0.0001 of the sum s.

A. 10 B. 300 C. 30 **D.** 100 E. 1000

---

21. The convergent power series in x that has the same equal to $\dfrac{x^3}{2-x^3}$ when $\|x\| < \sqrt[3]{2}$ is:

A. $\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^{3n+3}}{2^n}\right)$ B. $\sum_{n=0}^{\infty}\frac{x^{3n+3}}{2^n}$ C. $\sum_{n=0}^{\infty}\frac{x^{3n-3}}{4^n}$ D. $\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^{3n}}{2^n}\right)$ E. $\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^{3n+3}}{4^n}\right)$

---

22. Which of the following is the power series representation of $x\,e^{2x+1}$?

A. $x-2\cdot x^2+2^2\cdot\frac{x^3}{2!}-2^3\cdot\frac{x^4}{3!}+\cdots$

B. $e\cdot x^2+2\,e\cdot x^4+2^2\,e\cdot\frac{x^6}{2!}+2^3\,e\cdot\frac{x^8}{3!}+\cdots$

C. $e\cdot x+2\,e\cdot x^2+2^2\,e\cdot\frac{x^3}{2!}+2^3\,e\cdot\frac{x^4}{3!}+\cdots$

D. $x+(2x+1)\,x+\frac{(2x+1)^2}{2!}+\frac{(2x+1)^3}{3!}+\cdots$

E. $x^2+x^4+\frac{x^6}{2!}+\frac{x^8}{3!}+\cdots$

23. Which of the following series converge?

$$1)\sum_{n=1}^{\infty}\frac{1}{n} \qquad 2)\sum_{n=1}^{\infty}\frac{(-1)^n\cdot n}{\ln(n)} \qquad 3)\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

A. 1 B. 2 **C. 3** D. 2, 3 E. None

---

24. Which of the following is a power series representation for $f(x) = \dfrac{1}{4+6x}$ if $|x| < \dfrac{2}{3}$?

A. $\frac{1}{4}\left(\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{2}\right)^n x^n\right)$

B. $\frac{1}{6}\left(\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{3}\right)^n x^n\right)$

C. $\frac{1}{4}\left(\sum_{n=0}^{\infty}(-1)^{2n}\left(\frac{3}{4}\right)^n x^n\right)$

D. $\frac{1}{4}\left(\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^n x^n\right)$

E. $\frac{1}{6}\left(\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{2}\right)^n x^n\right)$

---

25. Which of the following is a power series representation for $f(x) = x\cdot(\arctan(x))$ if $|x| < 1$?

A. $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \cdots$

B. $1 - x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^9}{9} + \cdots$

C. $1 + x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \frac{x^{11}}{11} + \cdots$

D. $x^2 - \frac{x^4}{3} + \frac{x^6}{5} - \frac{x^8}{7} + \frac{x^{10}}{9} + \cdots$

E. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \cdots$

26. Which of the following is the power series representation of $\ln(1+2x)$?

A. $2x - \frac{4x^2}{2} + \frac{8x^3}{3} - \frac{16x^4}{4} + \frac{32x^5}{5} + \cdots$

B. $\frac{x^3}{3} - \frac{4x^4}{4} + \frac{8x^5}{5} - \frac{16x^6}{6} + \frac{32x^7}{7} + \cdots$

C. $2 - 4x + 8x^2 - 16x^3 + 32x^4 + \cdots$

D. $2x^2 - 4x^4 + 8x^6 - 16x^8 + 32x^{10} + \cdots$

E. $2x^2 - \frac{4x^4}{2} + \frac{8x^6}{3} - \frac{16x^8}{4} + \frac{32x^{10}}{5} + \cdots$

---

27. Which of the following is a power series representation for $f(x) = \frac{3}{(1+x)^2}$ if $|x| < 1$?

A. $2x^2 - 3x^3 + 4x^4 - 5x^5 + \cdots$

B. $3 - 3x + 3x^2 - 3x^3 + \cdots$

C. $1 - x + x^2 - x^3 + \cdots$

D. $3 - 3\cdot(2x) + 3\cdot(3x^2) - 3\cdot(4x^3) + \cdots$

E. $1 - 2x + 3x^2 - 4x^3 + \cdots$

---

28. Which of the following series converge?

1) $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^{3n}}$ 2) $\sum_{n=1}^{\infty} \frac{1}{(n+1)^3}$ 3) $\sum_{n=1}^{\infty} \frac{n+1}{\sqrt{n^3+2}}$

A. None B. 1 **C.** 2 D. 3 E. 2, 3

---

29. The Taylor Series about $x = 2$ for a certain function f converges to $f(x)$ for all x in the radius of convergence. The nth derivative of f at $x = 2$ is given by

$$f^{(n)}(x) = \frac{2^n\, n!}{3^{n+1}(n+1)}$$

and $f(2) = 3$. Which of the following is the radius of convergence for the Taylor series for f about $x = 2$?

A. $-\frac{1}{2}$ B. $\frac{2}{3}$ C. 0 **D.** $\frac{3}{2}$ E. 3

30. Let f be a function having derivatives of all orders for all real numbers x. The third-order Taylor polynomial for f about $x = 4$ is given by

$$T(x) = \frac{1}{9} + 5(x-4)^2 - 8(x-4)^3.$$

If $\left|f^{(4)}(x)\right| \leq \frac{1}{4}$ for $3.5 \leq x \leq 4$, order the following from greatest to least?

I. The maximum value of $|T(x) - f(x)|$ for $3.5 \leq x \leq 4$
II. $|f''(4)|$
III. $|f(4)|$

A. I, II, III B. II, I, III C. III, I, II **D. II, III, I** E. III, II, I

---

31. Suppose the second-degree Taylor polynomial for $f(x)$ around $x = 0$ is given by

$$P(x) = 7x + \frac{5}{2}x^2$$

If $g(x)$ is the inverse of $f(x)$, what is $g'(0)$?

A. 1 B. $\frac{1}{5}$ **C.** $\frac{1}{7}$ D. 5 E. Undefined

---

32. Find the limit of the sequence $a_n = 2 + \left(-\frac{4}{5}\right)^n$:

A. 2 B. $\frac{6}{5}$ C. $-\frac{4}{5}$ D. -2 E. $\frac{4}{5}$

---

33. The coefficient of $(x-3)^2$ in the Taylor series for $f(x) = \arctan(x)$ about $x = 3$ is:

A. $\frac{3}{100}$ **B.** $-\frac{3}{100}$ C. $-\frac{3}{50}$ D. $\frac{3}{50}$ E. $\frac{1}{256}$

Free Response

34. (10 points) Find the value of $\sum_{n=2}^{\infty} \frac{3^n + 5^n}{15^n}$. Show your work.

Solution:

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{3^n + 5^n}{15^n} &= \sum_{n=2}^{\infty} \frac{3^n}{15^n} + \frac{5^n}{15^n} \\
&= \sum_{n=2}^{\infty} \left(\frac{1}{5}\right)^n + \left(\frac{1}{3}\right)^n \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{n+1} + \left(\frac{1}{3}\right)^{n+1} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^2 \cdot \left(\frac{1}{5}\right)^{n-1} + \left(\frac{1}{3}\right)^2 \cdot \left(\frac{1}{3}\right)^{n-1} \\
&= \frac{1}{25} \cdot \frac{1}{1 - \frac{1}{5}} + \frac{1}{9} \cdot \frac{1}{1 - \frac{1}{3}} = \boxed{\frac{13}{60}}
\end{aligned}$$

35. (10 points) Show that the series $\sum_{n=2}^{\infty} \frac{1}{n \cdot (\ln(n))^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Solution: In this problem, we are asked to find values of p such that our given series converges. To this end, let

$$f(x) = \frac{1}{x \cdot (\ln(x))^p}.$$

For $x \geq 2$, we confirm that $f(x)$ is positive, decreasing and continuous using methods from Math 1A. Thus, by the integral test for convergence, we know that our given series and the integral

$$\int_2^{\infty} f(x)dx$$

have identical convergence behavior (the either both converge or they both diverge). With this in mind, consider the two cases:

I. Case $p = 1$:

$$\int_2^{\infty} f(x)dx = \int_2^{\infty} \frac{1}{x \cdot \ln(x)} dx$$

$$= \int_{\ln(2)}^{\infty} \frac{1}{u} du \qquad \text{Let } u = \ln(x) \Longrightarrow du = \frac{1}{x} dx \text{ and } u(2) = \ln(2) \text{ while } u(\infty) = \infty$$

$$= \lim_{t \to \infty} \int_{\ln(2)}^{t} \frac{1}{u} du$$

$$= \lim_{t \to \infty} \ln(u)\Big|_{\ln(2)}^{t}$$

$$= \left(\lim_{t \to \infty} \ln(t)\right) - \ln(2) = \infty.$$

Thus, for $p = 1$ our integral diverges. By the integral test for series, we conclude that the given series also diverges for $p = 1$.

II. Case $p \neq 1$:

$$\int_{2}^{\infty} f(x)dx = \int_{2}^{\infty} \frac{1}{x \cdot (\ln(x))^p} dx$$

$$= \int_{\ln(2)}^{\infty} \frac{1}{u^p}\, du$$

Let $u = \ln(x) \Longrightarrow du = \frac{1}{x} dx$ and

$u(2) = \ln(2)$ while $u(\infty) = \infty$

$$= \lim_{t\to\infty} \int_{\ln(2)}^{t} u^{-p}\, du$$

$$= \lim_{t\to\infty} \frac{u^{1-p}}{1-p}\Big|_{\ln(2)}^{t}$$

$$= \left(\lim_{t\to\infty} \frac{t^{1-p}}{1-p}\right) + c \text{ where } c = -\frac{(\ln(2))^{1-p}}{1-p}.$$

Thus we see that the integral converges if $p > 1$ and diverges for all $p \leq 1$. This is what was to be shown.

36. (10 points) Use a Taylor Polynomial to estimate the value of the integral

$$\int_0^{0.5} \ln(1+x^2)dx$$

with an absolute error of at most 1/1000. Justify your answer.

Solution: Recall from lesson 21 that the power series representation of the natural log function is given by

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{u^n}{n}$$

By Theorem 9.4.3 on page 679 concerning Combining Power Series via composition, we know that for $u = x^2$, we have

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^{2n}}{n}$$

In this problem, we are asked to find a definite integral over the interval $[0, 0.5]$ involving the function $\ln(1+x^2)$. We know by Theorem 9.5.2 on page 680, that when integrating the power series representation of this function, we can integrate the series term by term. Mathematically, we have

$$\int_0^{0.5} \ln(1+x^2)\ dx = \int_0^{0.5} \left[\sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^{2n}}{n} \right] dx$$

$$= \sum_{n=1}^{\infty} \left[\int_0^{0.5} (-1)^{n-1}\frac{x^{2n}}{n} \right] dx$$

$$= \sum_{n=1}^{\infty} \left[(-1)^{n-1} \cdot \frac{1}{n} \cdot \frac{x^{2n+1}}{2n+1} \Bigg|_0^{0.5} \right]$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n} \cdot \frac{1}{2n+1} \cdot \left[\frac{1}{2} \right]^{2n+1}$$

$$= \frac{(0.5)^3}{3} - \frac{(0.5)^5}{10} + \frac{(0.5)^7}{21} - \frac{(0.5)^9}{36} + \frac{(0.5)^{11}}{55} - \frac{(0.5)^{13}}{78} + \cdots$$

This is an alternating series and we can apply The Remainder Theorem 8.20 on page 652. Specifically, we see that since the third term

$$\frac{(0.5)^7}{21} \approx 0.000372 < 0.001 = \frac{1}{1000}$$

we know we can add the first to terms to approximate our integral to the desired accuracy. Calculating this approximation, we see

$$\int_0^{0.5} \ln(1+x^2)dx \approx \frac{(0.5)^3}{3} - \frac{(0.5)^5}{10} = 0.0385416\overline{6}.$$

Challenge Problem

37. (Optional, Extra Credit, Challenge Problem)

Suppose that a sequence $\{a_n\}_{n=1}^{\infty}$ satisfies

$$0 < a_n \leq a_{2n} + a_{2n+1}$$

for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Solution: Let us begin this problem by considering the significance of the stated property that $0 < a_n \leq a_{2n} + a_{2n+1}$ for our sequence $\{a_n\}_{n=1}^{\infty}$. Consider

$$\begin{aligned} a_1 &\leq a_2 + a_3 \\ a_2 + a_3 &\leq a_4 + a_5 + a_6 + a_7 \\ a_4 + a_5 + a_6 + a_7 &\leq a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} \end{aligned}$$

Notice we can find a pattern in the indices for these inequalities:

$$\begin{aligned} 1 &\leq 2 + 3 \\ 2 + 3 &\leq 4 + 5 + 6 + 7 \\ 4 + 5 + 6 + 7 &\leq 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 \end{aligned}$$

Let's use this pattern to generalize to the kth inequality. Set

$$p_k = \sum_{j=2^{k-1}}^{2^k - 1} a_j$$

and notice we can write $p_k \leq p_{k+1}$. Thus, we must have

$$\sum_{n=1}^{2^N - 1} a_j = \sum_{k=1}^{N} p_k \geq N p_1 = N a_1.$$

As $N \to \infty$ we see the partial sums are unbounded since $a_1 > 0$ and thus the original series must diverge as claimed.