True/False

For the problems below, circle T if the answer is true and circle F is the answer is false. After you've chosen your answer, mark the appropriate space on your Scantron form. Notice that letter A corresponds to true while letter B corresponds to false.

1. **T** F If f has a local minimum at point $(a, b) \in \mathbb{R}^2$, then

$$D_{\mathbf{u}f(a,b)} = 0$$

for any unit vector $\mathbf{u} \in \mathbb{R}^2$.

2. **T** F If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, then $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|_2 \, \|\mathbf{y}\|_2$

3. T **F** Suppose $f : \mathbb{R}^3 \to \mathbb{R}$. If $\nabla f = 0$ at a point $\mathbf{x} \in \mathbb{R}^3$, then f has a local extreme value at point $\mathbf{x}$.

4. **T** F For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$.

5. **T** F $f_y(a, b) = \lim\limits_{y \to b} \dfrac{f(a, y) - f(a, b)}{y - b}$.

6. T **F** The set of points $\{(x, y, z) : x^2 + y^2 = 1\}$ is a circle.

7. T **F** If $f(x, y) \to L$ as $(x, y) \to (a, b)$ along every straight line through (a, b),, then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L.$$

Multiple Choice

For the problems below, circle the correct response for each question. After you've chosen your answer, mark your answer on your Scantron form.

8. Let $z = \sin(x \cdot y)$ and let $x = x(t)$ and $y = y(t)$ be functions of t. Suppose

$$x(1) = 0, \qquad y(1) = 1, \qquad x'(1) = 2, \qquad y'(1) = 3.$$

Find $\dfrac{dz}{dt}$ when $t = 1$.

A. 1   **B.** 2   C. 3   D. 4   E. 5

9. Find the direction of maximum increase of the function $f(x, y, z) = x\,e^{-y} + 3\,z$ at the point $(1, 0, 4)$.

A. $\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$ **B.** $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ C. $\begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$ D. $\begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$ E. $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

10. Find the shortest distance from the origin to the surface $z^2 = 2xy + 2$

A. $\dfrac{1}{\sqrt{2}}$ **B.** $\sqrt{2}$ C. $\dfrac{1}{2}$ D. 2   E. 1

11. Find an equation for the line through the point $(3, -1, 2)$ and perpendicular to the plane $2x - y + z + 10 = 0$.

A. $3x - y + 2z + 10 = 0$ B. $3x - 2y + z + 10 = 0$ C. $\dfrac{x-2}{3} = \dfrac{y+1}{-1} = \dfrac{x-2}{2}$

D. $\dfrac{x-3}{2} = \dfrac{y+1}{-1} = z - 2$ E. $\dfrac{x+2}{2} = \dfrac{y-1}{-1} = z - 2$

12. Let $f(x, y) = e^{\sin(x)} + x^5 y + \ln\left(1 + y^2\right)$. Find f_{yx} :

A. $\dfrac{2y}{1+y^2}$ B. $20x^3 y$ **C.** $5x^4$ D. $e^{\sin(x)}\cos(x)$ E. $e^{\sin(x)}\cos(x) + x^5 + \dfrac{2y}{1+y^2}$

13. Find the area of the triangle with vertices at the points $(0, 0, 0)$, $(1, 0, -1)$ and $(1, -1, 2)$.

A. $\frac{\sqrt{11}}{2}$ B. $\frac{\sqrt{6}}{2}$ C. 1 D. $\sqrt{11}$ E. $\sqrt{6}$

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14. Given $f(x, y) = \sqrt{x^2 + y^2}$, find f_{xx} :

A. $\dfrac{y}{(x^2+y^2)^{1/2}}$ B. $\dfrac{xy}{(x^2+y^2)^{1/2}}$ C. $\dfrac{x}{(x^2+y^2)^{1/2}}$ D. $\dfrac{x^2}{(x^2+y^2)^{3/2}}$ **E.** $\dfrac{y^2}{(x^2+y^2)^{3/2}}$

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15. Find the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^4y^2}{x^4+3y^4}$

A. 0 B. 2 C. $\frac{1}{2}$ D. $\frac{2}{3}$ E. Does NOT exist

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16. Consider the vectors

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = 2\mathbf{i} + 3\mathbf{k}, \qquad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \mathbf{j} - \mathbf{k}.$$

Which of the following vectors gives $\mathbf{x} \times \mathbf{y}$?

A. $3\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ **B.** $-3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ C. $-3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ D. $-3\mathbf{k}$ E. -3

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17. Find an equation of the line through the point $(1, 2, 3)$ and parallel to the plane $x - y + z = 100$:

A. $x - y + z - 2 = 0$ B. $x - 1 = \frac{y+1}{2} = \frac{z-1}{3}$ C. $x + 2y + 3z = 100$

D. $x - 1 = 2 - y = z - 3$ E. $x - y + z + 2 = 0$

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18. The equation of the sphere with center $(4, -1, 3)$ and radius $\sqrt{5}$ is

A. $(x + 4)^2 + (y - 1)^2 + (z + 3)^2 = 5$

B. $(x - 4)^2 + (y - 1)^2 + (z - 3)^2 = 5$

C. $(x - 4)^2 + (y + 1)^2 + (z - 3)^2 = 25$

D. $(x - 4)^2 + (y + 1)^2 + (z - 3)^2 = \sqrt{5}$

E. $(x - 4)^2 + (y + 1)^2 + (z - 3)^2 = 5$

19. Given $\mathbf{x} = (2, 0, 1)$ and $\mathbf{v} = (4, 1, 2)$, what is the area of the parallelogram formed by the vectors $\mathbf{x}$ and $\mathbf{y}$?

A. $2\sqrt{5}$ **B.** $\sqrt{5}$ C. $2\sqrt{3}$ D. $3\sqrt{2}$ E. $4\sqrt{2}$

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20. Find values of $b \in \mathbb{R}$ such that the vectors $\begin{bmatrix} -11 \\ b \\ 2 \end{bmatrix}$ and $\begin{bmatrix} b \\ b^2 \\ b \end{bmatrix}$ are orthogonal.

A. $0, 3, -3$ B. $0, 11, -3$ C. $0, 2, -2$ D. $0, -11, 2$ E. $0, 11, 2$

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21. Find the parametric equations of the intersection of the planes $x - z = 0$ and $x - y + 2z + 3 = 0$

A. The line given by $x(t) = 1 + t, y(t) = 6$ and $z(t) = 1 - t$.
B. The line given by $x(t) = 1 + t, y(t) = 6 - t$ and $z(t) = 1 + 2t$.
C. The line given by $x(t) = -t, y(t) = 3 - 3t$ **and** $z(t) = -t$**.**
D. The plane $3x + 3y - 3z + 3 = 0$
E. The line given by $x(t) = -2 - t, y(t) = 1 - 3t$ and $z(t) = -t$.

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22. Given $\mathbf{x} = (4, 0)$ and $\mathbf{y} = (5, 2)$, which of the following is the projection of vector $\mathbf{x}$ onto the vector $\mathbf{y}$?

A. $\left(\frac{40}{27}, \frac{16}{27}\right)$ B. $(5, 0)$ C. $(4, 2)$ **D.** $\left(\frac{100}{29}, \frac{40}{29}\right)$ E. $\left(\frac{100}{\sqrt{29}}, \frac{40}{\sqrt{29}}\right)$

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23. Which of the following is a unit vector point in the direction of vector $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

A. $\frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ B. $\begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$ C. $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ **D.** $\frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ E. $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

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24. Find the distance between the point $(-1, -1, -1)$ and the plane $x + 2y + 2z - 1 = 0$

A. 6 **B. 2** C. 0 D. -2 E. -6

25. Consider the vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}.$$

Which of the following vectors gives $\mathbf{x} \cdot \mathbf{y}$?

A. 14 B. 10 **C.** -14 D. $-12\mathbf{k}$ E. -10

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26. Find the minimum value of the function $f(x, y) = x\,y$ subject to the constraint that $x^2 + y^2 = 2$:

A. 1 B. 2 **C.** -1 D. $\frac{3}{2}$ E. $-\frac{3}{2}$

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27. Find the directional derivative of the function

$$f(x, y) = y^2 \cdot \ln(x)$$

at the point $(1, 2)$ in the direction of the vector $(3, 4) = 3\mathbf{i} + 4\mathbf{j}$:

A. $\frac{5}{16}$ B. 12 C. $\frac{5}{12}$ D. $\frac{16}{5}$ **E.** $\frac{12}{5}$

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28. Find an equation of the tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 4$ at the point $(4, 1, 1)$.

A. $2x + y - z = 1$
B. $x + 2y + 2z = 8$
C. $x - 2y + 4z = 0$
D. $x + y + z = 6$
E. $2x + y + z = 10$

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29. Determine how many critical points the function $f(x, y) = x\,y - x^2\,y - x\,y^2$ has:

A. 1 B. 2 C. 3 **D.** 4 E. 5

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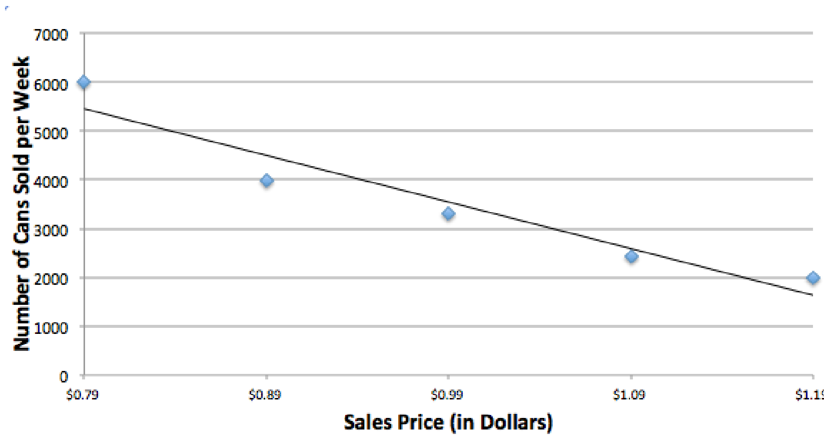
30. Let $f(x, y) = \frac{x}{y} + \frac{y}{x}$. Find the gradient vector ∇f:

A. $\begin{bmatrix} 2y \\ 2x \end{bmatrix}$ B. $\begin{bmatrix} x \\ y \end{bmatrix}$ **C.** $\begin{bmatrix} \frac{1}{y} - \frac{y}{x^2} \\ \frac{1}{x} - \frac{x}{y^2} \end{bmatrix}$ D. $\begin{bmatrix} -y/x^2 \\ -x/y^2 \end{bmatrix}$ E. $\begin{bmatrix} y \\ x \end{bmatrix}$

Free Response

31. (10 points) A company test-markets a new canned energy drink made of all natural ingredients in 5 cities of equal size on the West Coast of the US. The selling price (in dollars) and the number of drinks sold per week is each of the cities is listed as follows

City	Price	Sales/Week
1	0.79	6000
2	0.89	3980
3	0.99	3300
4	1.09	2440
5	1.19	1990



This data is plotted in the figure next to the table above. Although the data do not exactly lie on a straight line, we can create a linear model to fit this data.

A. Set up the least squares problem to fit this data to a linear model

$$S(p) = c_1 + c_2 p$$

where S is the sales per week and p is the price. Explain all choices that you made in setting up this model and describe why you made these choices.

Solution: Recall that the least squares problem is designed to fit data collected during an experiment to a particular mathematical model. In this case, we are told that our company collected five data points $\{(p_i, s_i)\}_{i=1}^5$, where

$$p_i = \text{the price per energy drink sold in city } i \text{ for } i = 1, 2, ..., 5$$
$$s_i = \text{the number of cans of the energy drink sold in city } i \text{ for } i = 1, 2, ..., 5$$

We notice that the model appears to fit a linear model $S(p) = c_1 + c_2 p$ for unknown parameters $c_1, c_2 \in \mathbb{R}$. This model can be used to predict the number of cans sold in city i based on the selling price p_i as follows:

$$S(p_i) = c_1 + c_2 \cdot p_i.$$

The difference between the observed data and the model prediction is known as the model error in the ith term, given by:

$$e_i = (S(p_i) - s_i) = (c_1 + c_2 \cdot p_i - s_i)\,.$$

To create the model of best fit for unknown parameters $c_1, c_2 \in \mathbb{R}$, we want to minimize the sum of the squared error terms:

$$f(c_1, c_2) = \sum_{i=1}^{5} e_i^2$$

$$= \sum_{i=1}^{5} (c_1 + c_2 \cdot p_i - s_i)^2$$

$$= (c_1 + 0.79 \cdot c_2 - 6000)^2 + (c_1 + 0.89 \cdot c_2 - 3980)^2$$

$$+ (c_1 + 0.99 \cdot c_2 - 3300)^2 + (c_1 + 1.09 \cdot c_2 - 2440)^2 + (c_1 + 1.19 \cdot c_2 - 1990)^2$$

Thus, the least squares problem is to minimize the function $f(c_1, c_2)$.

B. Explain how you would use multivariable calculus to find the line of best fit.

Solution: We apply multivariable calculus to solve this problem by recalling the second derivative test for the multivariable function $f(c_1, c_2)$. In particular, we know f has a local minimum if and only if

A. $\nabla f = \mathbf{0}$

B. $\dfrac{\partial f}{\partial c_1} \cdot \dfrac{\partial^2 f}{\partial c_2} - \left(\dfrac{\partial f}{\partial c_1 \partial c_2}\right)^2 < 0$ with $\dfrac{\partial^2 f}{\partial c_1^2} > 0$.

Thus, to find the local minimum of f using multivariable calculus, we need to find the critical points of this function and apply the second derivative test for multivariable function appropriately.

Remark (preview of coming attractions): There are two drawbacks of this method worth mentioning:

I. The method of minimizing the square of the modeled error is algebraically intensive. It requires us to expand the multivariable function $f(c_1, c_2)$ into quadratic terms in c_1 and c_2. Further to find the zeros of this polynomial requires non-linear methods.

II. Although multivariable calculus can be used to verify that the critical point where $\nabla f = \mathbf{0}$ is a local minimum, there is theoretical result that can conclude that this point will also be a global minimum. Thus, without further analysis of the function f, this method will not always guarantee a unique absolute minimum error term.

In Math 2B (Linear Algebra), we will revisit this problem using least squares techniques to improve the methods we discussed in this class.

32. (10 points) Let $f : \mathbb{R}^2 \to R$ be a differentiable, multivariable function. Let $\mathbf{u}$ be a unit vector.

A. Derive the dot product formula for the limit definition of the directional derivative $D_{\mathbf{u}}f$.

Solution: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a multivariable function with two input variables. Suppose $\mathbf{u} = a\mathbf{i} + b\mathbf{j} = (a, b)$ be a unit vector. To find the directional derivative of f at a point $\mathbf{x}_0 = (x_0, y_0) \in \mathbb{R}^2$ in the direction of $\mathbf{u}$, let $\mathbf{x} = \mathbf{x}_0 + t\,\mathbf{u}$ us consider the following limit:

$$D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|_2} = \lim_{t\to 0} \frac{f(x_0 + at, y_0 + bt) - f(x_0, y_0)}{t} = g'(0)$$

where we introduce the auxiliary function

$$g(t) = f(x_0 + a\,t, y_0 + b\,t).$$

For reference, the conversion from the first limit to the second follows from function evaluation at the vectors $\mathbf{x}$ and $\mathbf{x}_0$ along with the calculation:

$$\|\mathbf{x} - \mathbf{x}_0\|_2 = \|t\mathbf{u}\|_2 = |t|.$$

We can now use the auxiliary function $g(t)$ combined with the chain rule for multivariable functions to find an equivalent representation for $g'(t)$:

$$g'(t) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= f_x(x_0 + a\,t, y_0 + b\,t)a + f_y(x_0 + a\,t, y_0 + b\,t)b$$

Substituting the value $t = 0$ into this equation leads to

$$D_{\mathbf{u}}f(\mathbf{x}_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \nabla f \cdot \mathbf{u}.$$

B. Prove that the gradient vector is in the direction of maximum increase.

Solution: From Part A above, we know can use the dot product formula for the directional derivative of f in the direction of $\mathbf{u}$ to find

$$\begin{aligned} D_{\mathbf{u}}f &= \|\nabla f \cdot \mathbf{u}\|_2 \\ &= \|\nabla f\|_2 \; \|\mathbf{u}\|_2 \; \cos(\theta) \\ &= \|\nabla f\|_2 \; \cos(\theta) \\ &\leq \|\nabla f\|_2 \end{aligned}$$

We used the cosine formula for the dot product to get this series of equalities where θ is the angle between the gradient vector and the vector $\mathbf{u}$. Thus we see that the directional derivative is maximized when $\cos(\theta) = 1$, which happens when $\theta = 90°$. In other words, the gradient points in the direction of maximal increase.

33. Find the points on the surface defined by

$$(x-y)^2 + y^2 + (y+z)^2 = 1$$

at which the tangent plane is parallel to the xz-plane.

Solution: Let's define a three-variable function

$$f(x,y,z) = (x-y)^2 + y^2 + (y+z)^2.$$

In this case, we want to find the point(s) $(a,b,c) \in \mathbb{R}^3$ such that the tangent plane to the level surface $f(x,y,z) = 1$ has a normal vector in the direction of $\mathbf{n} = \langle 0,1,0 \rangle$. We know that the tangent plane to our level surface has equation

$$L(x,y,z) = f(a,b,c) + f_x(a,b,c)\,(x-a) + f_y(a,b,c)\,(y-b) + f_z(a,b,c)\,(z-c)$$

where the normal vector to this tangent plane is the vector $\nabla f = \langle f_x, f_y, f_z \rangle$. Now we want to find points on the surface such that

$$\nabla f(a,b,c) = \begin{bmatrix} f_x(a,b,c) \\ f_y(a,b,c) \\ f_z(a,b,c) \end{bmatrix} = \begin{bmatrix} 2a-2b \\ -2a+6b+2c \\ 2b+2c \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for some scalar $\lambda \in \mathbb{R}$. The reason that we introduce the scalar λ is because the only requirement we have is that the tangent plane should have a normal vector *in the direction* of $\mathbf{n}$, meaning it should be some scalar multiple times $\mathbf{n}$. Now, we can state our three equations that result from above:

$$\begin{aligned} &\text{Equation 1:} && 2a-2b = 0 \\ &\text{Equation 2:} && -2a+6b+2c = \lambda \\ &\text{Equation 3:} && 2b+2c = 0 \end{aligned}$$

Let's use elimination to solve for (a,b,c). First, we have

$$a = b \qquad \text{and} \qquad c = -b$$

by equations 1 and 3, respectively. We substitute both of these equalities into equation 2 to find

$$-2b+6b-2b = 2b = \lambda \qquad \Longrightarrow \qquad b = \frac{\lambda}{2}.$$

Any point on the surface with a tangent plane that is parallel to the $xz-$plane must be in the form

$$\left(\frac{\lambda}{2}, \frac{\lambda}{2}, -\frac{\lambda}{2}\right)$$

for some $\lambda \in \mathbb{R}$. However, we also know that any such point(s) also must satisfy the equation for the level surface

$$\left(\frac{\lambda}{2} - \frac{\lambda}{2}\right)^2 + \left(\frac{\lambda}{2}\right)^2 + \left(\frac{\lambda}{2} - \frac{\lambda}{2}\right)^2 = \left(\frac{\lambda}{2}\right)^2 = 1$$

We can simplify this equation as

$$\begin{aligned} \lambda^2 = 4 \qquad &\Longrightarrow \qquad \sqrt{\lambda^2} = 2 \\ &\Longrightarrow \qquad |\,\lambda\,| = 2. \end{aligned}$$

Since the values $\lambda_1 = 2$ and $\lambda_2 = -2$ both satisfy this equation, we know there are two points on our surface that have tangent planes parallel to the $xz-$plane. Moreover, these points are

$$\boxed{(1,1,-1)} \qquad \text{and} \qquad \boxed{(-1,-1,1)}$$

34. Let $z = z(x, y)$ be defined implicitly by equation

$$x^2 + y^2 - z^2 = 3xyz$$

Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at point $(3, 1, 1)$.

Solution: Let's check that point $(x, y, z) = (3, 1, 1)$ is on the surface described by the given equation:

$$x^2 + y^2 - z^2 = (3)^2 + 1^2 - 1^2 = 9 = 3 \cdot 3 \cdot 1 \cdot 1 = 3xyz$$

Next, we notice that the given equation gives an implicit relationship between the variables x, y and z. With a little thought, we see that it is algebraically very messy to try to solve for z explicitly in terms of x and y. Instead, let's apply the implicit differentiation technique we discussed in Lesson 10. To do so, we need to assume $z = z(x, y)$ is a differentiable function of x and y. Now, to find $\dfrac{\partial z}{\partial x}$, we apply the partial derivative with respect to x to both sides of our equation

$$\frac{\partial}{\partial x}\left[\; x^2 + y^2 - z^2 \;\right] = \frac{\partial}{\partial x}\left[\; 3xyz \;\right]$$

$$\Longrightarrow \quad \frac{\partial}{\partial x}\left[\; x^2 \;\right] + \frac{\partial}{\partial x}\left[\; y^2 \;\right] - \frac{\partial}{\partial x}\left[\; z^2 \;\right] = 3y\,\frac{\partial}{\partial x}\left[\; x \cdot z \;\right]$$

$$\Longrightarrow \quad 2x - 2z\,\frac{\partial z}{\partial x} = 3yz + 3xy\,\frac{\partial z}{\partial x}$$

The last line in this inequality comes from treating z as a function of x and applying the proper rules of differentiation including Chain Rule. Now, we can isolate the partial derivative of z with respect to x on one side of the equation to find

$$\left.\frac{\partial z}{\partial y}\right|_{(3,1,1)} = \left.\frac{2x - 3yz}{3xy + 2z}\right|_{(3,1,1)} = \boxed{\frac{3}{11}}$$

We find $\dfrac{\partial z}{\partial y}$ in a similar manner. We begin by taking the partial derivative with respect to y of both sides of our original equation and applying the same techniques as above

$$\frac{\partial}{\partial y}\left[\; x^2 + y^2 - z^2 \;\right] = \frac{\partial}{\partial y}\left[\; 3xyz \;\right]$$

$$\Longrightarrow \quad \frac{\partial}{\partial y}\left[\; x^2 \;\right] + \frac{\partial}{\partial y}\left[\; y^2 \;\right] - \frac{\partial}{\partial y}\left[\; z^2 \;\right] = 3x\,\frac{\partial}{\partial y}\left[\; y \cdot z \;\right]$$

$$\Longrightarrow \quad 2x - 2z\,\frac{\partial z}{\partial y} = 3yz + 3xy\,\frac{\partial z}{\partial y}$$

Again, we isolate the partial derivative of z with respect to y on one side of the equation to find

$$\left.\frac{\partial z}{\partial y}\right|_{(3,1,1)} = \left.\frac{2y - 3xz}{3xy + 2z}\right|_{(3,1,1)} = \boxed{-\frac{7}{11}}$$

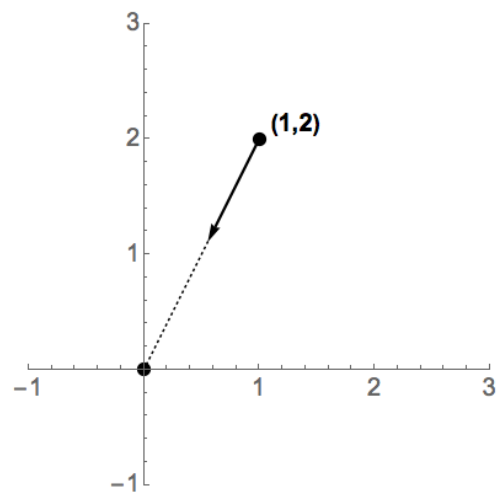
35. Consider the function $f(x, y) = 1 + x^2 + y^2$.

A. Find the equation for the tangent plane to $f(x, y)$ at point $(1, 2)$.

> **Solution:** The tangent plane to surface $z = f(x, y)$ at point (a, b) is given by equation
>
> $$L(x, y) = f(a, b) + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b)$$
>
> Since $(a, b) = (1, 2)$, we use our knowledge of partial derivatives to find
>
> $$f(1, 2) = 6, \qquad f_x(1, 2) = 2, \qquad f_y(1, 2) = 4.$$
>
> Then, we substitute these values into our general equation for $L(x, y)$ and do some arithmetic to find $L(x, y) = 2x + 4y - 4$

B. Use the tangent plane to approximate f as (x, y) moves a distance of $\dfrac{1}{10\sqrt{5}}$ units toward the origin.

> **Solution:** Now, we want to approximate the output value of $f(x, y)$ using our tangent plane by moving $\dfrac{1}{10\sqrt{5}}$ units toward the origin, starting at the point $(1, 2)$. Let's visualize this



> We begin by creating $\mathbf{v} = \overrightarrow{PO} = \langle -1, -2 \rangle$, which is a vector that points from point $P(1, 2)$ to origin $O(0, 0)$. We find the unit vector $\mathbf{u}$ in this direction by normalizing
>
> $$\mathbf{u} = \frac{\mathbf{v}}{\|\,\mathbf{v}\,\|_2} = \frac{\langle -1, -2 \rangle}{\sqrt{5}}.$$
>
> We are told that we want to move $\dfrac{1}{10\sqrt{5}}$ in the direction of $\mathbf{u}$, yielding the point
>
> $$(x, y) = \frac{1}{10\sqrt{5}} \cdot \frac{\langle -1, -2 \rangle}{\sqrt{5}} = \left(\frac{49}{50}, \frac{98}{50} \right)$$
>
> Then, using our tangent plane approximation, we see that
>
> $$f\left(\frac{49}{50}, \frac{98}{50} \right) \approx L\left(\frac{49}{50}, \frac{98}{50} \right) = \boxed{5.8}.$$

36. A. Use a scalar projection to show that the distance from point $P(x_1, y_1)$ to line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

Draw a diagram and explain your reasoning in detail using full sentences.

Solution:

Case 1: If $b = 0$, then we have a vertical line given by $x = \dfrac{-c}{a}$. The distance from the point $P(x_1, y_1)$ to this vertical line will be the length of the horizontal line segment that connects point P to this line. In other words, this distance will be

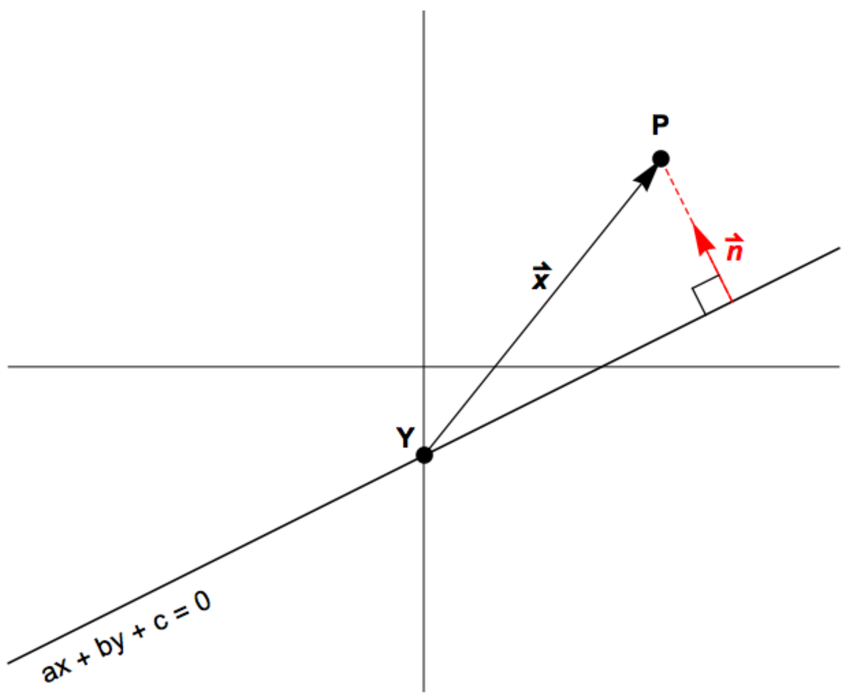
$$\left| x_1 - \frac{-c}{a} \right| = \left| \frac{ax_1 + c}{a} \right| = \frac{|ax_1 + c|}{\sqrt{a^2}}$$

The last equality comes from the fact that $|a| = \sqrt{a^2}$. But, this is exactly the formula we are asked to prove if $b = 0$.

Case 2: Assume $b \neq 0$. Consider the given equation for our line $ax + by + c = 0$. We can transform this equation into slope-intercept form using our knowledge of algebra. To this end, consider

$$ax + by = -c \qquad \Longrightarrow \qquad y = \frac{-a}{b}x + \frac{-c}{b}$$

The slope of this line is $m = \dfrac{-a}{b}$ and the $y-$ intercept is at the point $Y\left(0, \frac{-c}{b}\right)$. Let's graph this line on the cartesian plane, below, along with the point P.



By our discussion in Lesson 7, we know that the normal vector to this line is given by $\mathbf{n} = \langle a, b \rangle$. We construct the vector $\mathbf{x} = \overrightarrow{YP}$ that starts at the $y-$intercept Y and points toward the point P. This vector has coordinates

$$\mathbf{x} = \left\langle x_1, y_1 + \left(\frac{c}{b}\right) \right\rangle$$

To find the distance from point P to the given line $ax + by + c = 0$, we calculate the scalar component of $\mathbf{x}$ in the direction of $\mathbf{n}$. This is given by the equation

$$\mathrm{Scal}_{\mathbf{n}}(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{n}}{\| \mathbf{n} \|_2} = \frac{\left\langle x_1, y_1 + \left(\frac{c}{b}\right) \right\rangle \cdot \langle a, b \rangle}{\sqrt{a^2 + b^2}} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

This is the exact formula we wanted to derive.

B. Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

Solution: Now we apply the formula we derived in Part A. to the line $3x - 4y + 5 = 0$ and the point $P(-2, 3)$. Notice that we have

$$a = 3, \qquad b = -4, \qquad c = 5, \qquad x_1 = -2, \qquad y_1 = 3$$

With this, we see that the distance from the point P to the given line is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} = \boxed{\frac{13}{5}}$$