Exam 2: Extra Practice Problems

1. (FR) Find an equation to the tangent plane to the surface $4x^2 - y^2 + 3z^2 = 10$ at the point $(2, -3, 1)$.

Solution: Recall the equation for the linear approximation of a surface $z = f(x, y)$ at a point (a, b) given by

$$L(x, y) = f(a, b) + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b) \tag{1}$$

We are given $(a, b) = (2, -3)$. However, one of the major challenges in this problem is that we do not have z written as an explicit function of variables x and y. Instead, the given equation for the surface defines these values implicitly. Thus, in order to find f_x and f_y we have one of two choices:

i. Solve for z in terms of x and y

By manipulating the equation for our surface using algebra to isolate the z^2 term, we find that

$$z^2 = \frac{10 - 4x^2 + y^2}{3}$$

Since this relation does not specify a function (there are pairs of input values (x, y) that result in two different z values), we can partition this surface into two pieces:

$$f(x, y) = \sqrt{\frac{10 - 4x^2 + y^2}{3}} \qquad \text{and} \qquad g(x, y) = -\sqrt{\frac{10 - 4x^2 + y^2}{3}}.$$

Both of these functions describe the output z variable explicitly in terms of x and y. The problem statement explains that we want to expand our surface around the point $(2, -3, 1)$, which implies that we will use the $f(x, y) = \sqrt{10 - 4x^2 + y^2}$ description of our surface since $f(2, -3) = 1$ while $g(2, -3) = -1$. Now, we use our knowledge of partial differentiation to find

$$\frac{\partial f}{\partial x} = \frac{-4x}{3\sqrt{10 - 4x^2 + y^2}} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{y}{3\sqrt{10 - 4x^2 + y^2}}.$$

Using these descriptions, we see that $f_x(2, -3) = -8/3$ and $f_y(2, -3) = -1$. Substituting these values back into our linearization equation (1), we get our desired tangent plane

$$\boxed{L(x, y) = 1 - \frac{8}{3}(x - 2) - (y + 3).}$$

ii. Use implicit differentiation to find our desired partial derivatives

An alternative method to solving this problem involves implicit differentiation. In this method, we assume $z = z(x, y)$ is a "function" of x and y but allow the relationship to be described implicitly. Thus, to find the appropriate partial derivatives for our tangent plane approximation, we work to find z_x and z_y at our given point $(2, -3, 1)$ on the surface. To this end, consider:

$$\frac{\partial}{\partial x}\left[4x^2 - y^2 + 3z^2\right] = \frac{\partial}{\partial x}\left[10\right] \qquad \Longrightarrow \qquad -2y - 6z\frac{\partial z}{\partial x} = 0$$

$$\Longrightarrow \qquad \frac{\partial z}{\partial x} = -\frac{4x}{3z}$$

Substituting $x = 2$ and $z = 1$ into this equation, we see that $z_x = -8/3$. Similarly, we find

$$\frac{\partial}{\partial y}\left[4x^2 - y^2 + 3z^2\right] = \frac{\partial}{\partial y}\left[10\right] \qquad \Longrightarrow \qquad -2y + 6z\frac{\partial z}{\partial y} = 0$$

$$\Longrightarrow \qquad \frac{\partial z}{\partial y} = \frac{y}{3z}$$

Substituting $y = -3$ and $z = 1$ into this equation, we see $z_y = -1$. This results in the equation for the tangent plane

$$\boxed{L(x, y) = 1 - \frac{8}{3}(x - 2) - (y + 3).}$$

2. (MC) Consider the function $f(x, y) = 1 + x^2 + y^2$.

A. Find the equation for the tangent plane to $f(x, y)$ at point $(1, 2)$.

Solution: The tangent plane to surface $z = f(x, y)$ at point (a, b) is given by equation

$$L(x, y) = f(a, b) + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b)$$

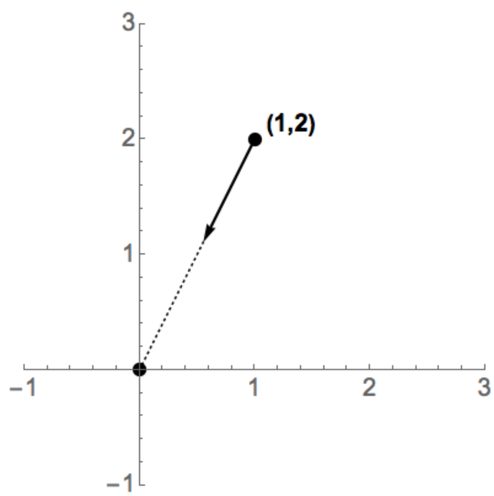
Since $(a, b) = (1, 2)$, we use our knowledge of partial derivatives to find

$$f(1, 2) = 6, \qquad f_x(1, 2) = 2, \qquad f_y(1, 2) = 4.$$

Then, we substitute these values into our general equation for $L(x, y)$ and do some arithmetic to find $L(x, y) = 2x + 4y - 4$

B. Use the tangent plane to approximate f as (x, y) moves a distance of $\frac{1}{10\sqrt{5}}$ units toward the origin.

Solution: Now, we want to approximate the output value of $f(x, y)$ using our tangent plane by moving $\frac{1}{10\sqrt{5}}$ units toward the origin, starting at the point $(1, 2)$. Let's visualize this



We begin by creating $\mathbf{v} = \overrightarrow{PO} = \langle -1, -2 \rangle$, which is a vector that points from point $P(1, 2)$ to origin $O(0, 0)$. We find the unit vector $\mathbf{u}$ in this direction by normalizing

$$\mathbf{u} = \frac{\mathbf{v}}{\|\,\mathbf{v}\,\|_2} = \frac{\langle -1, -2 \rangle}{\sqrt{5}}.$$

We are told that we want to move $\frac{1}{10\sqrt{5}}$ in the direction of $\mathbf{u}$, yielding the point

$$(x, y) = \frac{1}{10\sqrt{5}} \cdot \frac{\langle -1, -2 \rangle}{\sqrt{5}} = \left(\frac{49}{50}, \frac{98}{50}\right)$$

Then, using our tangent plane approximation, we see that

$$f\left(\frac{49}{50}, \frac{98}{50}\right) \approx L\left(\frac{49}{50}, \frac{98}{50}\right) = \boxed{5.8}.$$

3. (FR) Find the points on the surface defined by

$$(x-y)^2 + y^2 + (y+z)^2 = 1$$

at which the tangent plane is parallel to the xz-plane

Solution:

Let $f(x,y,z) = (x-y)^2 + y^2 + (y+z)^2 - 1$. We want to find points $(x_0, y_0, z_0) \in \mathbb{R}^3$ such that the tangent plane to function f is parallel to the $xz-$plane.

Recall, the vector equation for a plane is given by

$$0 = \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0)$$

where $\mathbf{n} = (a, b, c)$ is the normal vector to our plane, $\mathbf{r}_0 = (x_0, y_0, z_0)$ is a specific given point in our plane and $\mathbf{r} = (x, y, z)$ is the position of a general point on the plane.

We know that two planes are parallel if and only if their normal vectors are parallel. The normal vector to the $xz-$plane is given by $\mathbf{n} = (0, 1, 0)$. Thus, we want to find point(s) on the given surface where the tangent plane has normal vector $(0, 1, 0)$ or normal vector $(0, -1, 0)$. In the first case, we need to satisfy the following three conditions:

$$\begin{aligned} f_x(x,y,z) &= 2(x-y) = 0 \\ f_y(x,y,z) &= -2x + 6y + 2z = 1 \\ f_z(x,y,z) &= 2(y+z) = 0. \end{aligned}$$

We can solve this system of three equation with three unknowns using any method we'd like to find that the conditions hold if and only if $(x_0, y_0, z_0) = (0.5, 0.5, -0.5)$.

On the other hand, in the second case we need to have

$$\begin{aligned} f_x(x,y,z) &= 2(x-y) = 0 \\ f_y(x,y,z) &= -2x + 6y + 2z = -1 \\ f_z(x,y,z) &= 2(y+z) = 0. \end{aligned}$$

which occurs when $(x_0, y_0, z_0) = (-0.5, -0.5, 0.5)$. Thus, we have two different points on the surface where the tangent planes is parallel to the $xz-$plane.

4. (MC) Find all local extreme values and any saddle point(s) of the function $f(x, y) = 4xy - x^4 - y^4 + \frac{1}{16}$.

Solution: This problem is a multivariable optimization problem. To this end, we will use the multivariable second derivative test, restated below:

Theorem 12.14. *p. 941* ***Second Partial Derivatives Test***

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a two variable function. Suppose that $f(x, y)$ is twice differentiable on an open disk centered at the point (a, b) where $\nabla f(a, b) = \mathbf{0}$. Define the **discriminant** of f to be the function

$$D(x, y) = f_{xx}(x, y) \cdot f_{yy}(x, y) - (f_{xy}(x, y))^2$$

Then, we can use this function to make the following conclusions:

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b)
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b)
3. If $D(a, b) < 0$, then f has a saddle point at (a, b)
4. If $D(a, b) = 0$, then this test is inconclusive and cannot be used to identify the behavior of f at point (a, b)

We begin by finding all critical points where $\nabla f(x, y) = \mathbf{0}$. To this end, we consider

$$\nabla f(x, y) = \begin{bmatrix} 4y - 4x^3 \\ 4x - 4y^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving the resulting equations, we find the critical points happen when $y = x^3$ and $x = y^3$. There are three points in $\mathbb{R}^2$ where both of these equations hold simultaneously, given by

$$(-1, -1) \qquad (0, 0) \qquad (1, 1)$$

To apply the multivariable second derivative test, we check the sign of discriminant function

$$D = f_{xx} f_{yy} - f_{xy}^2 = 144\, x^2\, y^2 - 16$$

and the sign of $f_{xx} = -12x^2$ at each of these points in the table below:

Point	$D(x, y) = f_{xx} f_{yy} - f_{xy}^2$	f_{xx}	Classification of Critical Point using second partial derivative test
$(-1, -1)$	$D(-1, -1) = 128 > 0$	$f_{xx}(-1, -1) = -12 < 0$	Local Maximum
$(0, 0)$	$D(0, 0) = -16$	$f_{xx}(0, 0) = 0$	Saddle Point
$(1, 1)$	$D(1, 1) = 128 > 0$	$f_{xx}(1, 1) = -12 < 0$	Local Maximum

Using this table, we find all extreme values and saddle points of our function as was desired.

5. (MC) Find the extreme value(s) of the function $f(x, y) = 2x + 3y + 4$ on the circle $x^2 + y^2 = 1$

Solution: In this case, we have a constrained optimization problem. We know by our discussion in lecture that we can use Lagrange multipliers to solve this problem. To do so, we want to find a point (x, y) and scalar λ such that $\nabla f = \lambda \nabla g$. Consider:

$$\nabla f = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

This results in a set of three equations in three unknowns, given by

$$\begin{aligned} &\text{Equation 1:} && 2\,\lambda\,x = 2 \\ &\text{Equation 2:} && 3\,\lambda\,y = 3 \\ &\text{Equation 3:} && x^2 + y^2 = 1 \end{aligned}$$

Using equation 1, we can solve for λ in terms of x to find $\lambda = 1/x$. Substituting this value of lambda into equation 2, we can solve for y in terms of x to find $y = x$. Since this updated equation for y is strictly in terms of x, we can substitute this expression for y in equation 3 to find revised equation

$$2x^2 = 1$$

Then, we see that there are two points that satisfy each of these equations, given by

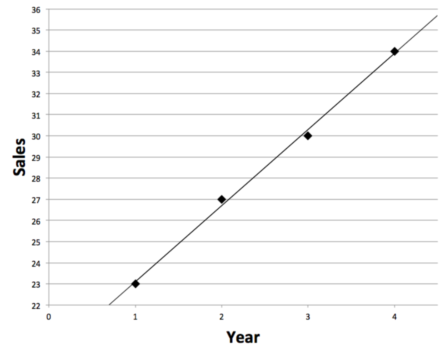
$$\left(\frac{1}{\sqrt{2}} \, , \, \frac{1}{\sqrt{2}} \right) \qquad\qquad \left(-\frac{1}{\sqrt{2}} \, , \, -\frac{1}{\sqrt{2}} \right)$$

Thus, on the curve $x^2 + y^2 = 1$, the function $f(x, y)$ has the following extreme values

$$\min_{x^2+y^2=1} f(x, y) = \qquad \text{and} \qquad \max_{x^2+y^2=1} f(x, y) =$$

6. (FR) A small bike company selling utility bicycles for daily commuting has been in business for four years. This company has recorded annual sales (in tens of thousands of dollars) as follows:

Year	Sales
1	23
2	27
3	30
4	34



This data is plotted in the figure next to the table above. Although the data do not exactly lie on a straight line, we can create a linear model to fit this data.

a. Set up the least squares problem to fit this data to a linear model.
b. Explicitly identify the unknown variables.

Solution: Recall that the least squares problem is designed to fit data collected during an experiment to a particular mathematical model. In this case, we are told that our company collects four data points $\{(t_i, s_i)\}_{i=1}^5$, where

$t_i =$ the ith year that the company has been in business for $i = 1, 2, 3, 4$

$s_i =$ the annual sales (in tens of thousands of dollars) during year i for $i = 1, 2, 3, 4$

We notice that the model appears to fit a linear model $S(t) = c_1 + c_2 t$ for unknown parameters $c_1, c_2 \in \mathbb{R}$. This model might be used to predict the revenues in i with:

$$S(p_i) = c_1 + c_2 \cdot t_i.$$

The difference between the observed data and the model prediction is known as the model error in the ith term, given by:

$$e_i = (S(t_i) - s_i) = (c_1 + c_2 \cdot t_i - s_i)\,.$$

To create the model of best fit for unknown parameters $c_1, c_2 \in \mathbb{R}$, we want to minimize the sum of the squared error terms:

$$\begin{aligned} f(c_1, c_2) &= \sum_{i=1}^{4} e_i^2 \\ &= \sum_{i=1}^{4} (c_1 + c_2 \cdot t_i - s_i)^2 \\ &= (c_1 + 1 \cdot c_2 - 23)^2 + (c_1 + 2 \cdot c_2 - 27)^2 \\ &\quad + (c_1 + 3 \cdot c_2 - 30)^2 + (c_1 + 4 \cdot c_2 - 34)^2 \end{aligned}$$

Thus, the least squares problem is to minimize the function $f(c_1, c_2)$.

c. Explain how you would use multivariable calculus to find the line of best fit.

d. What exactly is being optimized in the least squares problem?

Solution: We apply multivariable calculus to solve this problem by recalling the second derivative test for the multivariable function $f(c_1, c_2)$. In particular, we know f has a local minimum if and only if

A. $\nabla f = \mathbf{0}$

B. $\dfrac{\partial f}{\partial c_1} \cdot \dfrac{\partial^2 f}{\partial c_2} - \left(\dfrac{\partial f}{\partial c_1 \partial c_2}\right)^2 < 0$ with $\dfrac{\partial^2 f}{\partial c_1^2} > 0$.

Thus, to find the local minimum of f using multivariable calculus, we need to find the critical points of this function and apply the second derivative test for multivariable function appropriately.

Remark (preview of coming attractions): There are two drawbacks of this method worth mentioning:

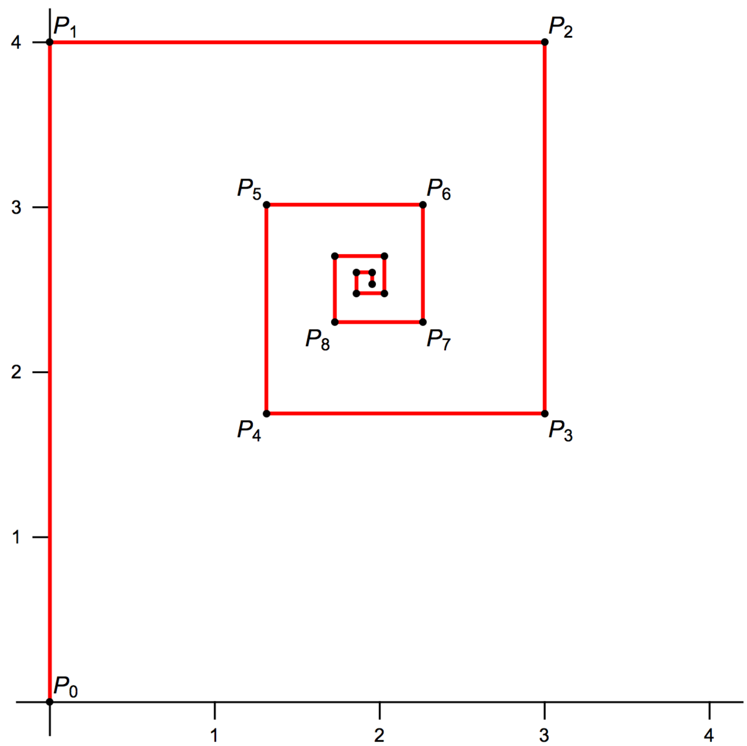
I. The method of minimizing the square of the modeled error is algebraically intensive. It requires us to expand the multivariable function $f(c_1, c_2)$ into quadratic terms in c_1 and c_2. Further to find the zeros of this polynomial requires non-linear methods.

II. Although multivariable calculus can be used to verify that the critical point where $\nabla f = \mathbf{0}$ is a local minimum, there is theoretical result that can conclude that this point will also be a global minimum. Thus, without further analysis of the function f, this method will not always guarantee a unique absolute minimum error term.

In Math 2B (Linear Algebra), we will revisit this problem using least squares techniques to improve the methods we discussed in this class.

7. (FR) Start at the origin and move 4 units along the positive y-axis. Turn 90 degrees to the right and move 75% of your last distance. Turn 90 degrees to the right and move 75% of your last distance. Turn 90 degrees to the right and move 75% of your last distance. Continue this processes forming a "spiral with square corners." Determine the **y-coordinate** for the point (x, y) where the spiral "ends."

Solution: We begin our solution by drawing a diagram of this process, as seen below:



Using this diagram and the problem statement, we can track the $y-$coordinate of each point in the spiral. The $y-$coordinate of each point on the square corners of this spiral changes every other move, due to the 90° shift. We see the $y-$coordinate of the "end" of the spiral is given by

$$y = 4 - 4 \cdot \left(\frac{3}{4}\right)^2 + 4 \cdot \left(\frac{3}{4}\right)^4 - 4 \cdot \left(\frac{3}{4}\right)^6 + 4 \cdot \left(\frac{3}{4}\right)^8 - 4 \cdot \left(\frac{3}{4}\right)^{10} + \cdots$$

$$= 4 \sum_{n=1}^{\infty} \left[-\frac{9}{16} \right]^{n-1}$$

$$= 4 \cdot \frac{1}{1 - \left(\frac{-9}{16}\right)} = \boxed{\frac{64}{25}}$$

The last line of this equality comes from applying the geometric series formula with $r = -9/16$ combined with arithmetic.

8. (EC) The following series converges. Determine the **exact** value of this infinite seres.

$$\frac{1}{1}+\frac{2}{2}-\frac{3}{2^2}+\frac{4}{2^3}+\frac{5}{2^4}-\frac{6}{2^5}+\frac{7}{2^6}+\frac{8}{2^7}-\frac{9}{2^8}+\cdots$$

Solution: Assuming this series converges, let's begin by rewriting this series in a different order

$$\left(\frac{1}{1}+\frac{4}{2^3}+\frac{7}{2^6}+\cdots\right)+\left(\frac{2}{2}+\frac{5}{2^4}+\frac{8}{2^7}+\cdots\right)-\left(\frac{3}{2^2}+\frac{6}{2^5}+\frac{9}{2^8}+\cdots\right)$$

Now, we have three different series that we want to analyze:

Series 1: $$\sum_{n=1}^{\infty}\frac{3n-2}{2^{3n-3}}=\left(\frac{1}{1}+\frac{4}{2^3}+\frac{7}{2^6}+\cdots\right)$$

Series 2: $$\sum_{n=1}^{\infty}\frac{3n-1}{2^{3n-2}}=\left(\frac{2}{2}+\frac{5}{2^4}+\frac{8}{2^7}+\cdots\right)$$

Series 3: $$\sum_{n=1}^{\infty}\frac{3n}{2^{3n-1}}=\left(\frac{3}{2^2}+\frac{6}{2^5}+\frac{9}{2^8}+\cdots\right)$$

Notice, if we can find the exact limit of the following series

$$\sum_{n=1}^{\infty}\frac{n}{2^{3n}}$$

then, we can use the algebraic properties of infinite series combined with the geometric series limit to manipulate Series 1 - 3 and find there exact limits. To this end, let's expand some of the terms of this series and rewrite the order as follows

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n}{2^{3n}} &= \frac{1}{2^3}+\frac{2}{2^6}+\frac{3}{2^9}+\frac{4}{2^{12}}+\cdots+\frac{n}{2^{3n}}+\cdots\\[2ex]
&= \frac{1}{2^3}+\frac{1}{2^6}+\frac{1}{2^9}+\frac{1}{2^{12}}+\cdots+\frac{1}{2^{3n}}+\cdots\\
&\qquad +\frac{1}{2^6}+\frac{1}{2^9}+\frac{1}{2^{12}}+\cdots+\frac{1}{2^{3n}}+\cdots\\
&\qquad\qquad +\frac{1}{2^9}+\frac{1}{2^{12}}+\cdots+\frac{1}{2^{3n}}+\cdots\\
&\qquad\qquad\qquad +\frac{1}{2^{12}}+\cdots+\frac{1}{2^{3n}}+\cdots\\
&\qquad\qquad\qquad\qquad \vdots\\
&\qquad\qquad\qquad\qquad +\frac{1}{2^{3n}}+\cdots\end{aligned}$$

We know by the geometric series test that

$$L=1+\frac{1}{2^3}+\frac{1}{2^6}+\frac{1}{2^9}+\frac{1}{2^{12}}+\cdots+=\sum_{n=1}^{\infty}\left[\frac{1}{2^3}\right]^{n-1}=\frac{1}{1-\frac{1}{8}}=\frac{8}{7}.$$

Using our expansion from above, we see we can rewrite our series

$$\sum_{n=1}^{\infty} \frac{n}{2^{3n}} = \frac{1}{2^3} \cdot L + \frac{1}{2^6} \cdot L + \frac{1}{2^9} \cdot L + \frac{1}{2^{12}} \cdot L + \cdots + \frac{1}{2^{3n}} \cdot L + \cdots = L \cdot \sum_{n=1}^{\infty} \frac{1}{2^{3n}}$$

Doing a little arithmetic on the last limit, we see

$$\sum_{n=1}^{\infty} \frac{1}{2^{3n}} = \frac{1}{2^3} \cdot \sum_{n=1}^{\infty} \left[\frac{1}{2^3}\right]^{n-1} = \frac{1}{2^3} \cdot L.$$

With this we have a closed form for our desired limit

$$\sum_{n=1}^{\infty} \frac{n}{2^{3n}} = \frac{1}{2^3} \cdot L^2 = \frac{1}{8} \cdot \frac{8^2}{7^2} = \frac{8}{49}.$$

Now, we can using this limit to find the exact values of series 1, 2, and 3. Let's begin with series 1:

Series 1:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3n-2}{2^{3n-3}} &= \left(\sum_{n=1}^{\infty} \frac{3n}{2^{3n-3}}\right) - \left(\sum_{n=1}^{\infty} \frac{2}{2^{3n-3}}\right) \\
&= \left(\frac{3}{2^{-3}} \sum_{n=1}^{\infty} \frac{n}{2^{3n}}\right) - \left(\sum_{n=1}^{\infty} 2 \cdot \left[\frac{1}{2^3}\right]^{n-1}\right) \\
&= \left(24 \cdot \frac{8}{49}\right) - \left(\frac{2}{1 - \frac{1}{8}}\right) \\
&= \frac{192}{49} - \frac{16}{7} = \boxed{\frac{80}{49}.}
\end{aligned}$$

We move on to the second series:

Series 2:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3n-1}{2^{3n-2}} &= \left(\sum_{n=1}^{\infty} \frac{3n}{2^{3n-2}}\right) - \left(\sum_{n=1}^{\infty} \frac{1}{2^{3n-2}}\right) \\
&= \left(\frac{3}{2^{-2}} \sum_{n=1}^{\infty} \frac{n}{2^{3n}}\right) - \left(\sum_{n=1}^{\infty} \frac{1}{2} \cdot \left[\frac{1}{2^3}\right]^{n-1}\right) \\
&= \left(12 \cdot \frac{8}{49}\right) - \left(\frac{1/2}{1 - \frac{1}{8}}\right) \\
&= \frac{96}{49} - \frac{4}{7} = \boxed{\frac{68}{49}.}
\end{aligned}$$

Finally, we end with our third series:

Series 3:
$$\sum_{n=1}^{\infty} \frac{3n}{2^{3n-1}} = \left(\sum_{n=1}^{\infty} \frac{3n}{2^{3n-1}} \right)$$
$$= \left(\frac{3}{2^{-1}} \sum_{n=1}^{\infty} \frac{n}{2^{3n}} \right)$$
$$= \left(6 \cdot \frac{8}{49} \right)$$
$$= \boxed{\frac{48}{49}.}$$

In the original problem statement, we had added series 1 and 2 and then subtracted series 3. Thus, the exact value of the sum in this problem is

$$\frac{80}{49} + \frac{68}{49} - \frac{48}{49} = \boxed{\frac{100}{49}}$$

9. (MC) Determine whether each of the following series converges or diverges. Write clear and complete solutions including the name of the series test that you use and what your final answer is.

(a) $\sum_{n=1}^{\infty} \frac{n+2}{3n+5}$

Solution: Let's define the sequence terms

$$a_n = \frac{n+2}{3n+5}$$

We notice that the sequence terms that define this series converge to $0.33\bar{3}$, since

$$\lim_{n\to\infty} \frac{n+2}{3n+5} = \lim_{n\to\infty} \left[\frac{n+2}{3n+5} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \right] = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{3+\frac{5}{n}} = \frac{1}{3}$$

We recall the Divergence Test, which states

Theorem 8.8. *p. 627* ***Divergence Test***

If the infinite series $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then the infinite series $\sum a_k$ diverges.

Thus, by the test for divergence we know that since $\lim_{n\to\infty} a_n \neq 0$, the corresponding series in our problem must diverge.

(b) $\sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n$

Solution: We notice that our sequence terms take the exact form of a geometric sequence, with a single ration raised to the nth power. With this in mind, we recall the Geometric Series Test, given as follows:

Theorem 8.7. *p. 621* ***Geometric Series Test***

Let $a \neq 0$ and let r be a real number. Then, the series

$$\sum_{k=1}^{\infty} ar^{k-1}$$

has the following convergence behavior:

If $|r| < 1$, then the series converges and $\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$.

If $|r| \geq 1$, then the series diverges.

Note: We can write the series using different upper and lower index as follows:

$$\sum_{k=1}^{\infty} ar^{k-1} = \sum_{k=0}^{\infty} ar^k$$

In both cases, we can determine the convergence behavior based on the geometric sum formula combined with our knowledge of the limits of geometric sequences.

If we set $a = -2/3$ and $r = -2/3$, then we can conclude that

$$\sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n = -\frac{2}{3} \cdot \left(\sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^{n-1} \right) = -\frac{2}{3} \cdot \frac{1}{1 - \frac{-2}{3}} = -\frac{2}{5}$$

Thus, since the infinite series has a finite limit, we know this series converges.

(c) $\sum_{n=2}^{\infty} \sqrt{\frac{n}{n^4+3}}$

Solution: Let's define the sequence terms

$$a_n = \sqrt{\frac{n}{n^4+3}}.$$

In this problem, we will attempt to bound each sequence term a_n above by some b_n such that $\sum_{n=2}^{\infty} b_n$ converges. If we can do this, we will then apply the direct comparison test which states

Theorem 8.15. *p. 643* ***The (Direct) Comparison Test***

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be infinite series with positive terms.

1. If $0 < a_k \leq b_k$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ converge, then $\sum_{k=1}^{\infty} a_k$ converges.
2. If $0 < b_k \leq a_k$ for all $k \in \mathbb{N}$ and $\sum_{k=1}^{\infty} b_k$ diverge, then $\sum_{k=1}^{\infty} a_k$ diverges.

To this end, notice the following sequence of inequalities for all $n \in \mathbb{N}$:

$$n^4 + 3 \geq n^4 \implies \frac{1}{n^4+3} \leq \frac{1}{n^4}$$

$$\implies \frac{n}{n^4+3} \leq \frac{n}{n^4} = \frac{1}{n^3}$$

$$\implies \sqrt{\frac{n}{n^4+3}} \leq \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$$

If we set $b_n = n^{-3/2}$, we see from above that $a_n \leq b_n$ for all $n \in \mathbb{N}$. We can make conclusions about the series defined by b_n using the $p-$series test, which states:

Theorem 8.11. *p. 632* ***Convergence of the p-Series (The p-Series Test)***

The ***p*-series** $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for all $p > 1$ and diverges for all $p \leq 1$.

Thus, by the p-series test, we have

$$\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$$

converges and we conclude that our original series also converges by the direct comparison test.

(d) $\sum_{n=3}^{\infty} \frac{(\ln(n))^2}{n}$

Solution: In this problem, let's use the integral test for series which is given as follows:

Theorem 8.10. *p. 629* ***Integral Test***

Suppose the function $f(x)$ satisfies the following three conditions for $x \geq 1$:

i. $f(x)$ is continuous
ii. $f(x)$ is positive
iii. $f(x)$ is decreasing

Suppose also that $a_k = f(k)$ for all $k \in \mathbb{N}$. Then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)dx$$

either both converge or both diverge. In the case of convergence, the value of the integral is NOT equal to the value of the series.

To this end, define the function

$$f(x) = \frac{(\ln(x))^2}{x}$$

on the interval $[3, \infty)$. We confirm that this function is positive, decreasing and continuous on this interval. By the integral test, we know that the series given in the problem statement converges if and only if the corresponding integral of our function $f(x)$ converges. Thus, let's consider the integral

$$\int_3^{\infty} f(x)dx = \int_3^{\infty} \frac{(\ln(x))^2}{x} dx \qquad \text{if } u = \ln(x), \text{ then } du = \frac{1}{x}dx$$

$$= \lim_{t\to\infty} \int_{x=3}^{x=t} u^2 du$$

$$= \lim_{t\to\infty} \frac{u^3}{3}\Big|_{x=3}^{x=t}$$

$$= \lim_{t\to\infty} \frac{(\ln(x))^3}{3}\Big|_{x=3}^{x=t}$$

$$= \lim_{t\to\infty} \frac{(\ln(t))^3}{3} - \frac{(\ln(3))^3}{3} = +\infty$$

Since our integral diverges, we know from the integral test that our given series also diverges.

(e) $\sum_{n=1}^{\infty} \frac{5^{n+1}}{(2n)!}$

Solution: We can define our sequence terms as

$$a_n = \frac{5^{n+1}}{(2n)!}$$

which contain an nth power and a factorial. Based on this structure, we recall that the ratio test may help us make a conclusion about the convergence behavior of this series. Let's recall the ratio test, given below:

Theorem 8.14. *p. 641* ***Ratio Test***

Let $\sum_{k=1}^{\infty} a_k$ be an infinite series with positive terms $a_k > 0$ for all $k \in \mathbb{N}$. Let

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$$

1. If $0 \leq r < 1$, then the series converges.
2. If $r > 1$ (including $r = \infty$), then the series diverges.
3. If $r = 1$, then the ratio test is inconclusive.

Note: In words, the ratio test says that the limit of the ratio of successive terms of a positive series must be less than 1 to guarantee convergence of the series.

With this in mind, consider

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{5^{n+2}}{(2n+2)!} \div \frac{5^{n+1}}{(2n)!} \\
&= \lim_{n\to\infty} \frac{5^{n+2}}{(2n+2)!} \cdot \frac{(2n)!}{5^{n+1}} \\
&= \lim_{n\to\infty} \frac{5}{(2n+1)\cdot(2n+2)} \\
&= \lim_{n\to\infty} \frac{5}{4n^2+4n+2} = 0.
\end{aligned}$$

Since the limit of the ratio above goes to zero as $n \to \infty$, we know by the ratio test that our series converges.

(f) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n+\sqrt{n}}$

Solution: In this problem, we set that the given series appears to be alternating. To this end, let's recall the alternating series test:

Theorem 8.18. *p. 650* ***Alternating Series Test***

Let $a_k > 0$ for all $k \in \mathbb{N}$ and consider the alternative series

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k$$

If we confirm BOTH of the following:

1. The terms of the series are nonincreasing in magnitude ($0 < a_{k+1} \leq a_k$ for k greater than some positive integer M)
2. $\lim_{k\to\infty} a_k = 0$

then the alternating series converges.

In this problem, let's define the sequence

$$a_n = \frac{1}{n+\sqrt{n}}.$$

Since both $n > 0$ and $\sqrt{n} > 0$ $n \in \mathbb{N}$, we see that all sequence terms a_n are positive. Next, we need to check our two conditions from the alternating series test:

1. We can confirm that the sequence terms are nonincreasing for all $n \in \mathbb{N}$ using the following chain of inequalities:

$$\begin{aligned} (n+1) \geq n \quad &\Longrightarrow \quad \sqrt{n+1} \geq \sqrt{n} \\ &\Longrightarrow \quad (n+1)+\sqrt{n+1} \geq n+\sqrt{n} \\ &\Longrightarrow \quad \frac{1}{(n+1)+\sqrt{n+1}} \leq \frac{1}{n+\sqrt{n}} \\ &\Longrightarrow \quad a_{n+1} \leq a_n \end{aligned}$$

   With this we confirm that $\{a_n\}_{n=1}^{\infty}$ is nonincreasing.

2. Next we consider

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+\sqrt{n}} = 0$$

   By the alternating series test, we the series given in this problem converges.

(g) $\frac{1}{1^4} + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} - \frac{1}{9^4} + \cdots$

Solution: Suppose that the series is define as

$$\sum_{k=1}^{\infty} a_k = \frac{1}{1^4} + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} - \frac{1}{9^4} + \cdots$$

for the appropriately chosen sequence $\{a_k\}_{k=1}^{\infty}$. If we define a new sequence $b_k = |a_k|$, then

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} |a_k|.$$

Recall that absolute convergence implies convergence, as is stated in the following theorem:

Theorem 8.21. *p. 651* ***Absolute Convergence Implies Convergence***

If $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

However, we see that the infinite series

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{n^4}$$

converges by the $p-$series test. Thus, our original series is absolutely convergent and thus, also converges.

(h) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$

Solution: We might be able to apply the telescoping sum technique to this problem. We begin by considering the sequence of partial sums associated with this infinite series, where

$$S_N = \sum_{n=1}^{N} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

Let's look at the first few sequence terms

$$S_1 \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}$$

$$S_2 = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} = 1 - \frac{1}{\sqrt{3}}$$

$$S_3 = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} = 1 - \frac{1}{\sqrt{4}}$$

Thus, we can find the limit of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{\sqrt{N+1}} = 1$$

Thus, we conclude that this series converges.

10. (MC) The alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+7}}$ converges. What should N be so that the partial sum

$$s_N = \sum_{n=1}^{N}(-1)^{n+1}\frac{1}{\sqrt{n+7}}$$

estimates the exact value of the series with absolute error at most 0.001?

Solution: In this problem, we are asked to estimate the value of the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+7}}, \qquad \text{where } a_n = \frac{1}{\sqrt{n+7}}.$$

Moreover, we want to produce an estimate with an absolute error no larger than 10^{-4}. To do so, we recall the Remainder Theorem for Alternating Series:

Theorem 8.20. *p. 652* ***Remainder in Alternating Series***

Let $S = \sum_{k=1}^{\infty}(-1)^{k+1}a_k$ be a convergent alternating series with terms that are nonincreasing in magnitude. Let $R_n = S - S_n$ be the remainder in approximating the value of the series by the sum of its first n terms. Then

$$|R_n| \le a_{n+1}$$

In other words, the magnitude of the remainder of a convergent alternating series. is less than or equal to the magnitude of the first neglected term.

With this in mind, we want to find $n \in \mathbb{N}$ such that

$$a_{n+1} < \frac{1}{10^4} \implies \frac{1}{\sqrt{(n+1)+7}} < \frac{1}{10^4}$$

$$\implies 10^4 < \sqrt{n+8}$$

$$\implies 10^8 < n+8$$

$$\implies 10^8 - 8 < n$$

Thus, if we set $N \ge 10^8$, we know that

$$|S - S_N| < 10^{-4}.$$

11. (MC) The series $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln(n))^2}$ converges. What should N be so that the partial sum

$$s_N = \sum_{n=1}^{N} \frac{1}{n(1+\ln(n))^2}$$

estimates the exact value of the series with absolute error at most 0.1?

Solution: In this problem, we are asked to estimate the value of the series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(1+\ln(n))^2}, \qquad \text{with } a_n = \frac{1}{n(1+\ln(n))^2}.$$

Moreover, we want to produce an estimate with absolute error is bounded above by 10^{-1}. We notice that if we define the function

$$f(x) = \frac{1}{x\,(1+\ln(x))^2},$$

then the sequence terms can be generated by evaluating this function at each natural number $a_n = f(n)$. Moreover, with a little analysis we see this function is positive, continuous, and decreasing on the interval $[1, \infty)$. Let's recall Integral Test Remainder Theorem:

Theorem 8.12. *p. 635* ***Estimating Series with Positive Terms***

Let $f(x)$ be a continuous, positive decreasing function, for $x \geq 1$, and define sequence $a_k = f(k)$ for all $k \in \mathbb{N}$. Suppose the limit of the associated convergent series is $S = \sum_{k=1}^{\infty} a_k$ and that the sequence of partial sums is $S_n = \sum_{k=1}^{n} a_k$. Then, the remainder $R_n = S - S_n$ satisfies the following inequality:

$$R_n < \int_n^{\infty} f(x)dx$$

In this problem we want to find N such that $R_N < 10^{-1}$. To this end, consider the integral

$$\int_N^{\infty} \frac{1}{x\,(1+\ln(x))^2}dx < \frac{1}{10} \implies \lim_{t\to\infty} \int_N^{t} \frac{1}{(1+u)^2}\,du < \frac{1}{10}$$

$$\implies \lim_{t\to\infty} \left. \frac{-1}{\ln(x)} \right|_N^t < \frac{1}{10}$$

$$\implies \lim_{t\to\infty} \frac{1}{\ln(N)} - \frac{1}{\ln(t)} < \frac{1}{10}$$

$$\implies \frac{1}{\ln(N)} < \frac{1}{10}$$

Then, in order to approximate our series so that our absolute error is less than e^{10}, we want to add $N > 10$ terms.