## Free Response

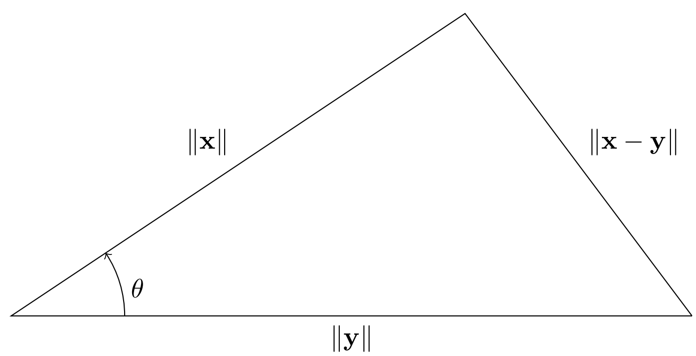
1. (8 points) Please explain your understanding of the dot product between two vectors in $\mathbb{R}^n$ below:

   A. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Derive the cosine formula for the dot product:

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|_2 \, \|\mathbf{y}\|_2 \cos(\theta)$$

You don't have to prove the pythagorean theorem nor the law of cosines in this derivation. However, please explain your work and specifically identify the steps you took to arrive at your final answer.

**Solution:** Case I: Assume $\mathbf{x}$ and $\mathbf{y}$ are not scalar multiples of each other. Suppose we begin with two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Consider the triangle defined by these vectors. The length of each side of this triangle can be given by the $2-$norm of the vectors:




By the Law of Cosines, we know

$$\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\|\mathbf{x}\|\,\|\mathbf{y}\|\,\cos(\theta)$$

Recall, using the algebraic properties of the inner product, we can write

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\|^2 &= (\mathbf{x}-\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y}) \\ &= \mathbf{x} \cdot (\mathbf{x}-\mathbf{y}) - \mathbf{y} \cdot (\mathbf{x}-\mathbf{y}) \\ &= \mathbf{x}\cdot\mathbf{x} - \mathbf{x}\cdot\mathbf{y} - \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} \\ &= \|\mathbf{x}\|^2 - 2\,\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2\end{aligned}$$

With this we see

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\|\mathbf{x}\|\,\|\mathbf{y}\|\,\cos(\theta) = \|\mathbf{x}\|^2 - 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2$$

By canceling out the appropriate terms using our knowledge of arithmetic, we see

$$\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|\,\cos(\theta).$$

Case II: Assume $\mathbf{x}$ and $\mathbf{y}$ are scalar multiples of each other (i.e. $\mathbf{y} = a\mathbf{x}$). In this case we know that the angle between our vectors is either $\theta = 0$ or $\theta = \pi$. If $\theta = 0$, then $a > 0$ and $\cos(\theta) = 1$. On the other hand if $\theta = \pi$, then $a < 0$ and $\cos(\theta) = -1$. In either case, we see

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot (a\,\mathbf{y})$$

$$= a\,\mathbf{x} \cdot \mathbf{x}$$

$$= a\|\mathbf{x}\|^2$$

$$= a\|\mathbf{x}\|\,\|\mathbf{x}\|$$

Recall that the sign function $f(x) = \operatorname{sgn}(x)$ is a piecewise function defined as follows:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Then, for any scalar $a \in \mathbb{R}$, we can write $a = \operatorname{sgn}(a)\,|a|$. Moreover, since $|a| = \sqrt{a^2}$, we see

$$\mathbf{x} \cdot \mathbf{y} = \operatorname{sign}(a)\,\sqrt{a^2}\,\sqrt{\sum_{i=1}^{n} x_i^2}\,\|\mathbf{x}\|_2$$

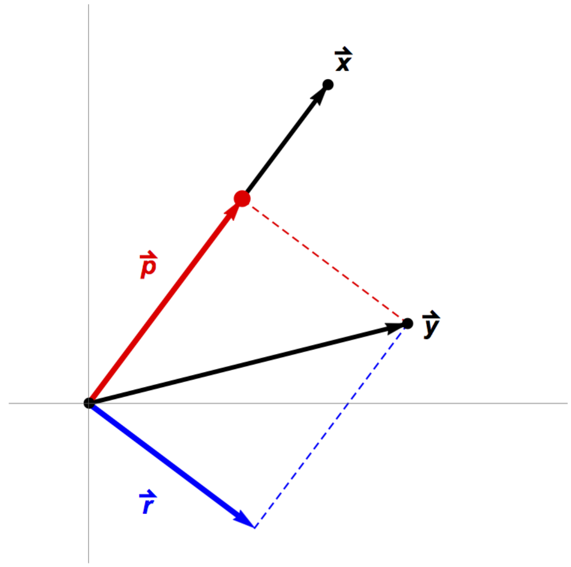
$$= \operatorname{sign}(a)\,\sqrt{\sum_{i=1}^{n} (ax_i)^2}\,\|\mathbf{x}\|_2$$

$$= \operatorname{sign}(a)\,\|\mathbf{y}\|_2\,\|\mathbf{x}\|_2$$

$$= \cos(\theta)\,\|\mathbf{y}\|_2\,\|\mathbf{x}\|_2$$

Thus we see that the cosine formula for the inner product holds

B. Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. Using the diagram below, derive an equation for the projection of vector $\mathbf{y}$ onto $\mathbf{x}$. Be sure to identify the projection vector $\mathbf{p}$ and the residual vector $\mathbf{r}$ on the diagram below. Please explain your answer and specifically identify the steps you took to arrive at your final answer.




**Solution:** In this problem, we want to find vector $\mathbf{p} = \text{Proj}_{\mathbf{x}}(\mathbf{y})$ the orthogonal projections of a vector $\mathbf{y} \in \mathbb{R}^3$ onto a vector $\mathbf{x} \in \mathbb{R}^3$. Here, we are asked to find the red vector $\mathbf{p}$ that is in the direction of $\mathbf{x}$ and represents an orthogonal projection of $\mathbf{y}$ onto $\mathbf{x}$. Since $\mathbf{p} \in \text{Span}(\mathbf{x})$, we know that $\mathbf{p} = \alpha\mathbf{x}$ for some scalar $\alpha \in \mathbb{R}$. If we can find a closed-form equation for the $\alpha$, we can construct our vector $\mathbf{p}$. To this end, let's rewrite our equation using the properties of vector operations with

$$\mathbf{r} = \mathbf{y} - \mathbf{p}.$$

By the cosine formula for the dot product, scalar $\alpha$ must satisfy the condition

$$0 = \mathbf{x} \cdot \mathbf{r} = \mathbf{x} \cdot (\mathbf{y} - \alpha\,\mathbf{x}) = \mathbf{x} \cdot \mathbf{y} - \alpha\,\mathbf{x} \cdot \mathbf{x}$$

The third equality results from the algebraic properties of dot products. Solving for $\alpha$, we find

$$\alpha = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{x} \cdot \mathbf{x}} = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|_2^2}$$

This produces our desired projection. We can now use algebra to rewrite the projection:

$$\mathbf{p} = \alpha\,\mathbf{x} = \left(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|_2^2}\right)\mathbf{x} = \left(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|_2}\right)\frac{\mathbf{x}}{\|\mathbf{x}\|_2}$$

This gives a systematic approach to creating a projection onto any axis we desire.

2. (8 points) Find the distance from the point $P(1,-2,4)$ to the plane $3x+2y+6z=5$. Please explain your answer and specifically identify the steps you took to arrive at your final answer.

**Solution:** We begin by finding an arbitrary point on the plane (any point that satisfies the equation for the plane). Let's choose point $Q(1,1,0)$. Then we create vector

$$\vec{\mathbf{x}} = \overrightarrow{PQ} = \langle 0\,,3\,,-4\rangle$$

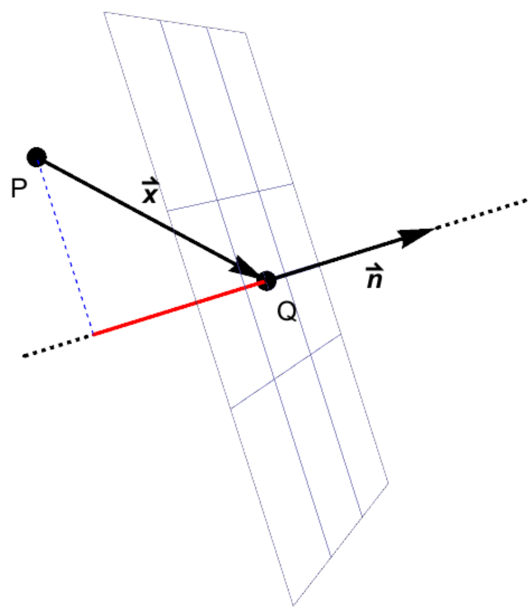
that connects point $P(1,-2,4)$ to the point $Q(1,1,0)$ on our plane. Recall also the dot product equation for our plane, given by

$$\vec{\mathbf{n}}\cdot(\mathbf{r}-\mathbf{r}_0)=0$$

where $\vec{\mathbf{n}}$ is the normal vector to our plane, $\mathbf{r}_0$ is a specific known point on our plane and $\mathbf{r}$ is a general point on our plane. We know that the coefficients in front of each variable from our scalar equation for the plane define the corresponding entries of this normal vector and we conclude

$$\mathbf{n} = \langle 3\,,2\,,6\rangle\,.$$

Then, the distance from point $P(1,-2,4)$ to the plane is the length of $\text{Proj}_{\vec{\mathbf{n}}}(\vec{\mathbf{x}})$. This is the so called "scalar projection of $\vec{\mathbf{x}}$ onto $\vec{\mathbf{n}}$." Below we draw a diagram to help describe this problem so far.




In this case, we are looking for the length of the red segment. We calculate this distance as

$$\begin{aligned} d &= \left\|\text{Proj}_{\vec{\mathbf{n}}}(\vec{\mathbf{x}})\right\|_2 \\ &= \left|\frac{\vec{\mathbf{x}}\cdot\vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|_2}\right| \\ &= \left|-\frac{18}{7}\right| = \boxed{\frac{18}{7} = 2\,\frac{4}{7} \approx 2.57143} \end{aligned}$$

$$\vec{\mathbf{p}}\cdot\vec{\mathbf{x}} = \begin{bmatrix}0\\3\\-4\end{bmatrix}\cdot\begin{bmatrix}3\\2\\6\end{bmatrix} = 0\cdot 3+3\cdot 3+(-4)\cdot 6 = -18$$

$$\|\vec{\mathbf{n}}\|_2 = \sqrt{3^2+2^2+6^2} = \sqrt{49} = 7$$

3. (8 points) For vectors $\mathbf{x} = \langle 1, 1, 2 \rangle$ and $\mathbf{y} = \langle -2, 3, 1 \rangle$, express the vector $\mathbf{x}$ as the sum of two vectors $\mathbf{x} = \mathbf{p} + \mathbf{r}$ where $\mathbf{p}$ is parallel to $\mathbf{y}$ and $\mathbf{r}$ is orthogonal to $\mathbf{y}$. Please explain your answer and specifically identify the steps you took to arrive at your final answer.

**Solution:** We see that this problem can be solved using the techniques we developed in our work in problem 1B. Specifically, we want to find vector $\hat{\mathbf{p}} = \text{Proj}_{\mathbf{y}}(\mathbf{x})$ the orthogonal projections of a vector $\mathbf{x} \in \mathbb{R}^3$ onto a vector $\mathbf{y} \in \mathbb{R}^3$. The figure below encapsulates the problem of orthogonal projection. This gives a systematic approach to creating a projection onto any axis we desire. Using our derivation in problem 1B, we see that the projection of $\vec{\mathbf{x}}$ onto $\vec{\mathbf{y}}$ is given by

$$\text{Proj}_{\vec{\mathbf{y}}}(\vec{\mathbf{x}}) = \left[ \frac{\vec{\mathbf{x}} \cdot \vec{\mathbf{y}}}{\|\vec{\mathbf{y}}\|_2^2} \right] \vec{\mathbf{y}} \qquad \vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = 1 \cdot (-2) + 1 \cdot 3 + 2 \cdot 1 = 3$$

$$\|\vec{\mathbf{y}}\|_2^2 = (-2)^2 + 3^2 + 1^2 = 14$$

$$= \frac{3}{14} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \boxed{\left\langle -\frac{6}{14}, \frac{9}{14}, \frac{3}{14} \right\rangle}$$

**HINT:** Be sure to spend extra time and care to fully understand the derivation of the projection formula above. This formula shows up in Linear Algebra as a fundamental idea that drives many of the algorithms we currently use in a wide variety of applications.

4. (6 points) Find two unit vectors orthogonal to both $\langle 3, 2, 1\rangle$ and $\langle -1, 1, 0\rangle$. Please explain your answer and specifically identify the steps you took to arrive at your final answer.

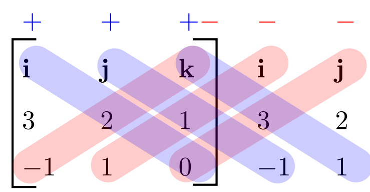
**Solution:** Recall that the cross product operation is design to produce an output vector that is orthogonal to the two given input vectors. We can find a unique "direction" orthogonal to both input vectors by finding the cross product $\mathbf{n}$ of these vectors. To this end, let's set

$$\mathbf{x} = \langle\, 3,\ 2,\ 1\,\rangle\,, \qquad\qquad \mathbf{y} = \langle\, -1,\ 1,\ 0\,\rangle\,.$$

Then, the vector $\mathbf{n}$ points in the direction orthogonal to both $\mathbf{x}$ and $\mathbf{y}$. In mathematical terms, we conclude that our desired direction is any scalar multiple of the vector given by the cross product

$$\mathbf{n} = \mathbf{x} \times \mathbf{y} = \langle -1\,, -1\,, 5\rangle\,.$$

where we calculated this cross product using the determinant formula we discussed in class:




Now, in this problem, we are asked to find the two unit vectors in this direction. To this end, we find the length of this vector

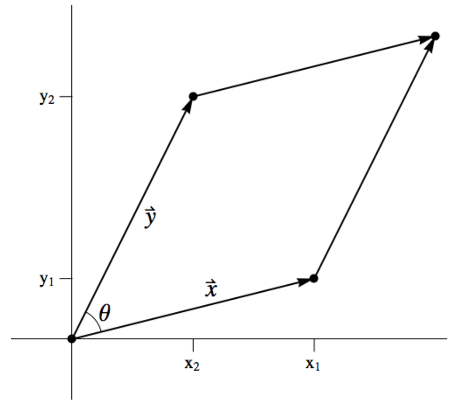
$$\|\mathbf{n}\|_2 = 3\sqrt{3}$$

We use this magnitude to normalize $\mathbf{n}$ and find two unit vectors orthogonal to $\mathbf{x}$ and $\mathbf{y}$:

$$\mathbf{u}_1 = \frac{\mathbf{n}}{\|\mathbf{n}\|_2} = \frac{1}{3\sqrt{3}} \cdot \langle -1\,, -1\,, 5\rangle\,, \qquad \text{and} \qquad \mathbf{u}_2 = -\frac{\mathbf{n}}{\|\mathbf{n}\|_2} = \frac{1}{3\sqrt{3}} \cdot \langle 1\,, 1\,, -5\rangle\,.$$

5. (12 points) Below, please explain your understanding of the cross product between two vectors in $\mathbb{R}^3$.

A. Let $\mathbf{x} = \langle x_1, y_1 \rangle$ and $\mathbf{y} = \langle x_2, y_2 \rangle$ be two vectors in $\mathbb{R}^2$. Using the diagram below, derive an equation for the area of the parallelogram formed by vectors $\mathbf{x}$ and $\mathbf{y}$ based only on the components of these vectors (note: this equation should NOT be based on the angle $\theta$ between these vectors). Please explain your answer and specifically identify the steps you took to arrive at your final answer.




B. Under the same assumptions in problem A. ,suppose that $\theta$ is the angle between above with $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. Derive a formula for the area of the parallelogram parallelogram formed by vectors $\mathbf{x}$ and $\mathbf{y}$ as a function of $\theta$ and the two norms of these vectors. Please explain your answer and specifically identify the steps you took to arrive at your final answer.

C. How do your answers to parts A. and B. on the last page relate to the component form of the cross product of the vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ where

$$\mathbf{a} = \langle a_1, b_1, c_1 \rangle, \qquad \mathbf{b} = \langle a_2, b_2, c_2 \rangle,$$

Make sure to explicitly state the component form of the cross product in your explanation.

D. Using the assumptions from part C above, derive the sine formula for the cross product:

$$\|\mathbf{a} \times \mathbf{b}\|_2 = \|\mathbf{a}\|_2 \, \|\mathbf{b}\|_2 \, |\sin(\theta)|$$

6. (8 points) Find the parametric equation for a line in the intersection of the planes $x + y + z = 1$ and $x - 2y + 3z = 1$. Please explain your answer and specifically identify the steps you took to arrive at your final answer.

**Solution:** We know that any line in $\mathbb{R}^3$ is defined by a point on the line and a direction vector. Let's recall the parametric equation for a line in $\mathbb{R}^3$, given by

$$\vec{\mathbf{r}}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \mathbf{r}_0 + t \cdot \mathbf{v} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

where $\mathbf{r}_0 \in \mathbb{R}^3$ is any point on our line and the vector $\mathbf{v} \in \mathbb{R}^3$ represents the direction of the line. In order to find the equation of a line in the intersection of two planes, we need to find a point $\mathbf{r}_0$ on both planes and the direction of our line of intersection.

We begin by finding a point on both planes. To this end, we need to find a point $\mathbf{r}_0$ that satisfies both plane equations given in our problem statement. If we assume $z_0 = 0$, we get a system of two equations in two unknowns given as follows:

$$x + y = 1,$$
$$x - 2y = 1.$$

Solving this equation, we see that $x_0 = 1$ and $y_0 = 0$. Hence, the point

$$\mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is on both planes (we can check that this point satisfies both plane equations as we claimed).

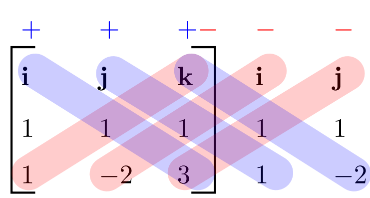
Next, we find the direction vector $\mathbf{v}$ of the line of intersection between both planes. This direction vector must sit in both planes. Notices that vectors

$$\mathbf{n}_1 = \langle 1 \,, 1 \,, 1 \rangle \,, \qquad \mathbf{n}_2 = \langle 1 \,, -2 \,, 3 \rangle \,.$$

are the normal vectors to the first and second plane, respectively. The direction vector $\mathbf{v}$ for our line should be orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$. In mathematical terms, we conclude that our desired direction vector can be given by the cross product between our normal vectors,

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 5 \,, -2 \,, -3 \rangle \,..$$

where we calculated this cross product using the determinant formula we discussed in class:




Now we can state the vector equation for the line of intersection between our two planes:

$$\vec{\mathbf{r}}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 + 5t \\ -2t \\ -3t \end{bmatrix}$$

## Challenge Problem

7. (Optional, Extra Credit, Challenge Problem) Derive the general equation for an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

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| **Solution:** Enter here. |

  