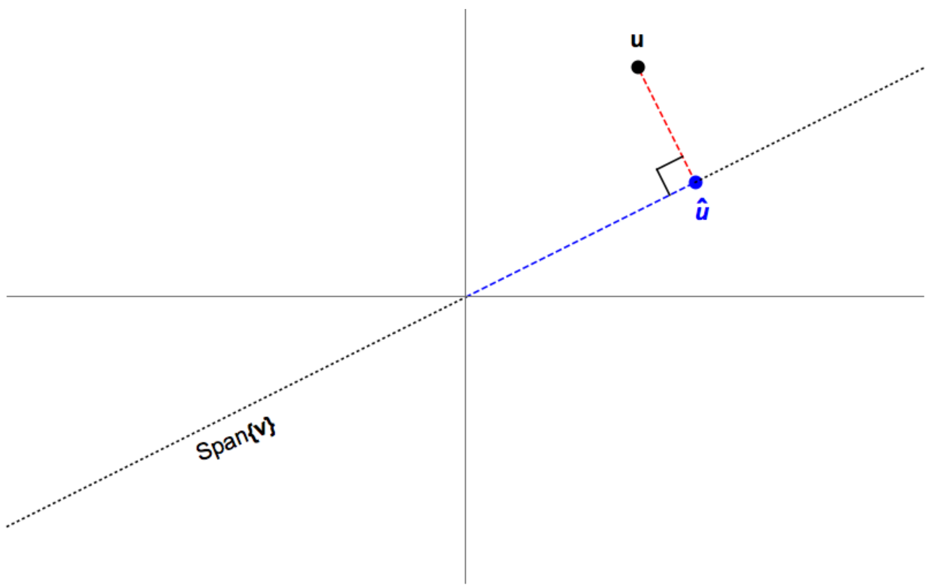
Exam 1: Practice Problems

1. (MC) Let $\mathbf{u} = \langle 1, -2 \rangle$ and $\mathbf{v} = \langle 3, 4 \rangle$. Find the $\text{Proj}_{\mathbf{v}}(\mathbf{u})$.

Solution:

In this problem, we want to find vector $\hat{\mathbf{u}} = \text{Proj}_{\mathbf{v}}(\mathbf{u})$ the orthogonal projections of a vector $\mathbf{u} \in \mathbb{R}^2$ onto a vector $\mathbf{v} \in \mathbb{R}^2$. The figure below encapsulates the problem of orthogonal projection.



Here, we are asked to find the blue vector $\hat{\mathbf{u}}$ that is in the direction of $\mathbf{v}$ and represents an orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$. Since $\hat{\mathbf{u}} \in \text{Span}(\mathbf{v})$, we know that $\hat{\mathbf{u}} = \alpha\mathbf{v}$ for some scalar $\alpha \in \mathbb{R}$. If we can find a closed-form equation for the α, we can construct our vector $\hat{\mathbf{u}}$. To this end, let's rewrite our equation using the properties of vector operations with

$$\mathbf{z} = \mathbf{u} - \hat{\mathbf{u}}.$$

By the cosine formula for the dot product, scalar α must satisfy the condition

$$0 = \mathbf{v} \cdot \mathbf{z} = \mathbf{v} \cdot (\mathbf{u} - \alpha\,\mathbf{v}) = \mathbf{v} \cdot \mathbf{u} - \alpha\,\mathbf{v} \cdot \mathbf{v}$$

The third equality results from the algebraic properties of dot products. Solving for α, we find

$$\alpha = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|_2^2}$$

This produces our desired projection. First, we consider the projection of $\mathbf{v}$ onto $\text{Span}\{\mathbf{y}\}$, given by

$$\hat{\mathbf{u}} = \alpha\,\mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|_2^2}\right)\mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|_2}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|_2}$$

This gives a systematic approach to creating a projection onto any axis we desire. Using this derivation, we see that the projection of $\vec{\mathbf{u}}$ onto $\vec{\mathbf{v}}$ is given by

$$\text{Proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}}) = \left[\frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|_2^2} \right] \vec{\mathbf{v}} \qquad \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 1 \cdot 3 + (-2) \cdot 4 = -5$$

$$\|\vec{\mathbf{v}}\|_2^2 = 3^2 + 4^2 = 25$$

$$= \left[\frac{-5}{25} \right] \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \boxed{\left\langle -\frac{3}{5}, -\frac{4}{5} \right\rangle}$$

WARNING: Be sure to spend extra time and care to fully understand the derivation of the projection formula above. This formula shows up in Linear Algebra as a fundamental idea that drives many of the algorithms we currently use in a wide variety of applications.

2. (MC) Define two lines $L_1(t)$ and $L_2(s)$ intersect at a single point in $\mathbb{R}^3$, where

$$\mathbf{L}_1(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1+t \\ 2t \\ -1+3t \end{bmatrix} \qquad \text{and} \qquad \mathbf{L}_2(s) = \begin{bmatrix} x(s) \\ y(s) \\ z(s) \end{bmatrix} = \begin{bmatrix} 3+2s \\ 1+s \\ -2-s \end{bmatrix}$$

Find the point (x, y, z) of intersection. Then, find the angle θ between the lines.

Solution:

Recall that two lines in $\mathbb{R}^3$ intersect at as single point. Thus, the lines given above intersect if and only if we can find parameters t and s such that

$$x(t) = x(s), \qquad y(t) = y(s), \qquad z(t) = z(s).$$

This gives us three equations in two unknowns. For the sake of simplicity, let's focus on the first two of these equations and solve for t and s accordingly. To this end, we see $x(t) = x(s)$ and $y(t) = y(s)$ if and only if

$$\begin{aligned} 2t &= 1+s \\ -1+3t &= -2-s \end{aligned}$$

We can solve this system of two equations in two unknowns using any method we desire. For the sake of completeness, let's use elimination. By our first equation, we see $s = 2t - 1$. We substitute this value into the second equation

$$-1+3t = -2-(2t-1)$$

We can solve this equation for t to find that when $t = 0$ and thus we must have $s = -1$. With this, we see that our two lines intersect at point

$$\boxed{(1\,, 0\,, -1)}$$

Now, we want to find the angle θ between these two lines. If we assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are the direction vectors for lines $\mathbf{L}_1(t)$ and $\mathbf{L}_2(t)$, respectively, we see from our problem statement that

$$\mathbf{v}_1 = \langle 1\,, 2\,, 3 \rangle\,, \qquad \mathbf{v}_2 = \langle 2\,, 1\,, -1 \rangle\,.$$

We can find the angle θ between these lines using the cosine formula for the dot product, with

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\|_2 \cdot \|\mathbf{v}_1\|_2 \cos(\theta)$$

Using cosine inverse, we see

$$\theta = \arccos\left(\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\|_2 \cdot \|\mathbf{v}_1\|_2} \right) = \arccos\left(\frac{1}{2\sqrt{21}} \right) \approx \boxed{1.46147 \text{ radians}}$$

3. (MC) Compute the distance from the origin $(0,0,0)$ to the plane $2x + y - 2z = 6$.

Solution: We begin by finding an arbitrary point on the plane (any point that satisfies the equation for the plane). Let's choose point P$(3,0,0)$. Then we create vector

$$\vec{\mathbf{p}} = \overrightarrow{OP}$$

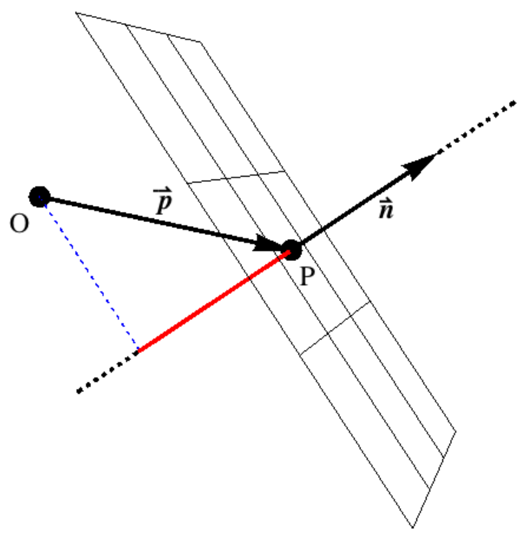
that connects the origin O$(0,0,0)$ to the point P$(3,0,0)$ on our plane. Recall also the dot product equation for our plane, given by

$$\vec{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

where $\vec{\mathbf{n}}$ is the normal vector to our plane. We know that the coefficients in front of each variable from our scalar equation for the plane define the corresponding entries of this normal vector and we conclude

$$\mathbf{n} = \langle 2\,, 1\,, -2 \rangle\,.$$

Then, the distance from point O$(0,0,0)$ to the plane is the length of $\text{Proj}_{\vec{\mathbf{n}}}(\vec{\mathbf{p}})$. This is the so called "scalar projection of $\vec{\mathbf{p}}$ onto $\vec{\mathbf{n}}$." Below we draw a diagram to help describe this problem so far.



In this case, we are looking for the length of the red segment. We calculate this distance as

$$d = \left\| \text{Proj}_{\vec{\mathbf{n}}}(\vec{\mathbf{p}}) \right\|_2 \qquad \vec{\mathbf{p}} \cdot \vec{\mathbf{n}} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 3 \cdot 2 + 0 \cdot 1 + 0 \cdot (-2) = 6$$

$$= \left| \frac{\vec{\mathbf{p}} \cdot \vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|_2} \right| \qquad \|\vec{\mathbf{n}}\|_2 = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$$

$$= \left| \frac{6}{3} \right| = \boxed{2}$$

4. (MC) Find a parametric representation of the line L in the intersection of the planes $x + 2z = 1$ and $x + y - z = 0$.

Solution: We know that any line in $\mathbb{R}^3$ is defined by a point on the line and a direction vector. Let's recall the parametric equation for a line in $\mathbb{R}^3$, given by

$$\vec{\mathbf{r}}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \mathbf{r}_0 + t \cdot \mathbf{v} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

where $\mathbf{r}_0 \in \mathbb{R}^3$ is any point on our line and the vector $\mathbf{v} \in \mathbb{R}^3$ represents the direction of the line. In order to find the equation of a line in the intersection of two planes, we need to find a point $\mathbf{r}_0$ on both planes and the direction of our line of intersection.

We begin by finding a point on both planes. To this end, we need to find a point $\mathbf{r}_0$ that satisfies both plane equations given in our problem statement. If we assume $z_0 = 0$, then the first equation forces $x_0 = 1$. Then, we can use the second equation to find that $y_0 = -1$. Hence, the point

$$\mathbf{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

is on both planes (we can check that this point satisfies both plane equations as we claimed).

Next, we find the direction vector $\mathbf{v}$ of the line of intersection between both planes. This direction vector must sit in both planes. Notices that vectors

$$\mathbf{n}_1 = \langle 1\,, 2\,, 0 \rangle\,, \qquad \mathbf{n}_2 = \langle 1\,, 1\,, -1 \rangle\,.$$

are the normal vectors to the first and second plane, respectively. The direction vector $\mathbf{v}$ for our line should be orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$. In mathematical terms, we conclude that our desired direction vector can be given by the cross product between our normal vectors,

$$\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle -2\,, 1\,, -1 \rangle\,..$$

where we calculated this cross product using the determinant formula we discussed in class:

$$\begin{array}{ccccc} + & + & +\;- & - & - \\ \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 1 & 2 & 0 & 1 & 2 \\ 1 & 1 & -1 & 1 & 1 \end{array}$$

Now we can state the vector equation for the line of intersection between our two planes:

$$\vec{\mathbf{r}}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

5. Let $\vec{\mathbf{A}} = \langle 3, 0, -2\rangle$ and $\vec{\mathbf{B}} = \langle 0, -1, 1\rangle$.

   a. (MC) Find the area of the parallelogram formed by placing the vectors tail-to-tail.

Solution: We calculate the entries of $\mathbf{n}$ using the determinant formula for the cross product

$$\begin{array}{ccccc} + & + & +\,- & - & - \\ \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 3 & 0 & -2 & 3 & 0 \\ 0 & -1 & 1 & 0 & -1 \end{array}$$

and we find the normal vector

$$\mathbf{n} = \vec{\mathbf{A}} \times \vec{\mathbf{B}} = \langle -2, -3, -3\rangle$$

By the sine formula for the dot product, we know

$$\|\mathbf{n}\|_2 = \|\vec{\mathbf{A}}\|_2 \cdot \|\vec{\mathbf{B}}\|_2 \cdot \sin(\theta),$$

where θ is the angle between vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. We see that $\|\mathbf{n}\|_2$ gives area of the parallelogram formed by placing the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ tail-to-tail. Thus, the area of our parallelogram is

$$\|\mathbf{n}\|_2 = \sqrt{(-2)^2 + (-3)^2 + (-3)^2} = \sqrt{4+9+9} = \boxed{\sqrt{22}}$$

   b. (MC) Find an equation of the plain containing point $(1, -2, 3)$ and which is parallel to both vectors.

Solution: Recall also the dot product equation for a plane is given by

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

where $\mathbf{n}$ is the normal vector to our plane and point is given in the problem statement as

$$\mathbf{r}_0 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

By construction, the vector $\mathbf{n} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is orthogonal to both vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Thus, the vector equation for any point $\langle x, y, z\rangle$ in our plane is given by

$$0 = \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = \begin{bmatrix} -2 \\ -3 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y+2 \\ z-3 \end{bmatrix}$$

We simplify using the definition of the dot product and the algebraic properties of $\mathbb{R}$ to find the scalar equation for this plane, which is given by

$$\boxed{2x + 3y + 3z = 5}$$

6. (MC) Let $z = 3x + \ln(x^2 + y)$. Compute the partial derivatives z_y, z_x and z_{xx}.

Solution: We begin by finding z_y given by

$$z_y = \frac{\partial}{\partial y}\Big[3x + \ln(x^2 + y)\Big] = \boxed{\frac{1}{x^2 + y}}$$

Next, we find z_x given by

$$z_x = \frac{\partial}{\partial x}\Big[3x + \ln(x^2 + y)\Big] = \boxed{3 + \frac{2x}{x^2 + y}}$$

Finally, we calculate z_{xx}, the second partial derivative of z with respect to x, given by

$$z_{xx} = \frac{\partial}{\partial x}\Big[z_x\Big] = \frac{\partial}{\partial x}\left[3 + \frac{2x}{x^2 + y}\right] = \boxed{\frac{-2x^2 + 2y}{(x^2 + y)^2}}.$$

7. (MC) Find the point(s) on the following function $f(x, y) = x^3 - y^3 + 3xy$ where both $f_x = 0$ and $f_y = 0$.

Solution: First we find the partial derivatives f_x and f_y using our derivative rules

$$f_x(x, y) = \frac{\partial}{\partial x}\Big[x^3 - y^3 + 3xy\Big] = 3x^2 + 3y = 3(x^2 + y)$$

$$f_y(x, y) = \frac{\partial}{\partial y}\Big[x^3 - y^3 + 3xy\Big] = -3y^2 + 3x = 3(x - y^2)$$

Now, we set both of these partial derivatives equal to zero

$$3(x^2 + y) = 0, \qquad\qquad 3(x - y^2) = 0.$$

To find any point(s) where these are both equal to zero, we see that both

$$y = -x^2 \qquad \text{and} \qquad x = y^2.$$

We can graph these two curves to notice that they intersect at two points in $\mathbb{R}^2$. In other words, there are two points that satisfy both of these equations simultaneously. Thus, the two points (x, y) that guarantee that $f_x = 0$ and $f_y = 0$ are

$$\boxed{(0, 0)} \qquad \text{and} \qquad \boxed{(1, -1)}$$

8. (MC) Evaluate the following limit: $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{1-\sqrt{x+y+1}}{x+y}$.

Solution: To evaluate this limit, we recall from Math 1A the algebraic method of multiplying by a conjugate:

$$\begin{aligned}
\lim_{(x,y)\to(1,-1)} \frac{1-\sqrt{x+y+1}}{x+y} &= \lim_{(x,y)\to(1,-1)} \frac{1-\sqrt{x+y+1}}{x+y} \cdot \frac{1+\sqrt{x+y+1}}{1+\sqrt{x+y+1}} \\
&= \lim_{(x,y)\to(1,-1)} \frac{-(x+y)}{(x+y)\cdot\left(1+\sqrt{x+y+1}\right)} \\
&= \left[\lim_{(x,y)\to(1,-1)} -\frac{(x+y)}{(x+y)}\right]\cdot\left[\lim_{(x,y)\to(1,-1)} \frac{1}{\left(1+\sqrt{x+y+1}\right)}\right] \\
&= -1\cdot\left[\frac{1}{1+1}\right] = \boxed{-\frac{1}{2}}.
\end{aligned}$$

9. (MC) Verify that the following limit does not exists: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$.

Solution: Recall that a limit in $\mathbb{R}^2$ exists if and only if the output of function $f(x,y)$ approaches L as input variables (x,y) approach point (a,b) from ANY direction. With this in mind, a quick way to check for nonexistence of a limit is the "two path test:"

> If $f(x,y)$ has a different limit along two different paths that approach (a,b) in the domain of f, then we know that $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Our goal then is to find two paths with different limiting behavior.

Path 1: Let's first move along the line $y=0$. To this end, consider:

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$$

Path 2: Next, let's move along the line $x=y^3$ and consider:

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{(y^3,y)\to(0,0)} \frac{y^3\cdot y^3}{\left(y^3\right)^2+y^6} = \lim_{y\to 0} \frac{y^6}{y^6+y^6} = \frac{1}{2}.$$

Since there are two paths that approach $(a,b)=(0,0)$ in the domain that result in different limiting behavior in the range, we know that this limit does not exist, as was to be shown.

10. (MC) Let $f(x, y) = 4 - \sqrt{y - x^2}$

(a) Determine and sketch the domain of f in 2D-space.

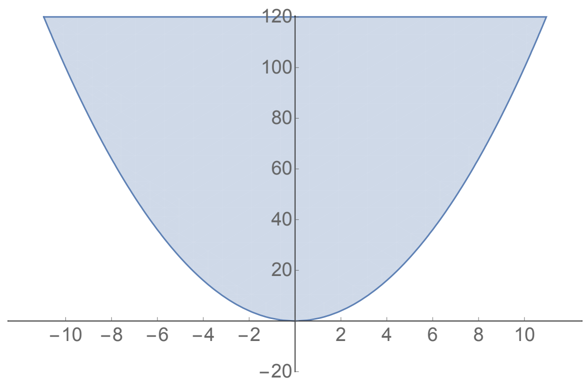
Solution: Notice that the expression $y - x^2$ inside the square root determines the domain of this two-variable function. In particular, since the argument of a square root must be nonnegative, we see that

$$y - x^2 \geq 0 \qquad \Longrightarrow \qquad y \geq x^2$$

Then, the domain of the function $f(x, y)$ defined in this problem is the set

$$\text{Dom}(f) = \{(x, y) \in \mathbb{R}^2 \, : \, y \geq x^2\}$$

This includes all points on or above the parabola $y = x^2$, as seen in the region below:



(b) State the range of f. BRIEFLY explain how you get your answer.

Solution: Consider the expression $\sqrt{y - x^2}$. We know that $x^2 \geq 0$ for any real value of x. Thus, the output of this square root is maximum when $x = 0$. Moreover, $0 \leq \sqrt{y} < \infty$. Hence, along the vertical line $x = 0$, the function

$$f(0, y) = 4 - \sqrt{y}$$

has a maximum value of 4 and is unbounded below. Because this must be a global maximum, we see

$$\text{Rng}(f) = (-\infty, 4]$$