16/07/2018

MIC, Lesson 6, Pt. 1 : In-Class Problems

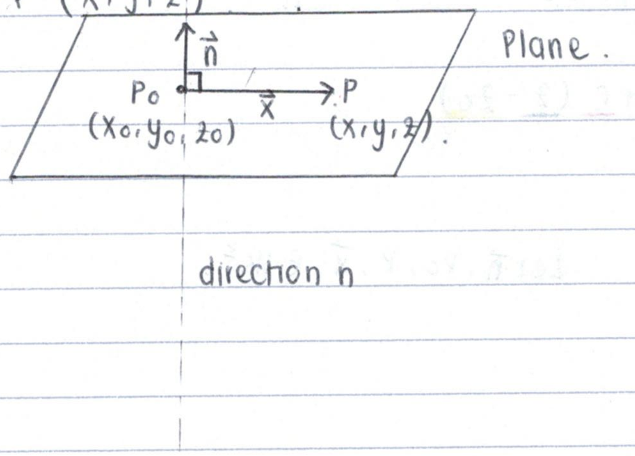
$\boxed{1}$. $n = \langle a, b, c \rangle$ normal vector

point P_0 (x_0, y_0, z_0)

Derive equation for a plane in $\mathbb{R}^3$.

a) diagram of a plane with P_0, n, $x = \overrightarrow{P_0P}$

Let P (x, y, z)



b).

Recall, orthogonality theorem for the dot product.

Given 2 vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$ orthogonal to one another.

$\vec{a} \cdot \vec{b} = 0$ perpendicular

In our case

Let $\vec{x} = \overrightarrow{P_0P}$

$\vec{n}$ & $\vec{x}$ are orthogonal to one another

Hence, perpendicular.

$\vec{n} \cdot \vec{x} = 0$

$\vec{n} \cdot \overrightarrow{P_0P} = 0$

Recall point P_0 (x_0, y_0, z_0) and $P(x, y, z)$

Hence $\vec{x} = \overrightarrow{P_0P}$

$= P - P_0$

$= (x, y, z) - (x_0, y_0, z_0)$

$= \langle x - x_0, y - y_0, z - z_0 \rangle$

Now let

$\vec{n} = \langle a, b, c \rangle$

Substituting this into

$\vec{n} \cdot \vec{x} = 0$

We get

$\vec{n} \cdot \vec{x} = 0$

$0 = \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$

$0 = a(x - x_0) + b(y - y_0) + c(z - z_0)$

c) Scalar equation for plane

Recall from part b)

$0 = a(x - x_0) + b(y - y_0) + c(z - z_0)$

where

$\vec{n} = \langle a, b, c \rangle$

↑

normal vector

Let $\vec{n}, P_0, P, \vec{x} \in \mathbb{R}^3$

Point $P\ (x, y, z)$

Point $P_0\ (x_0, y_0, z_0)$

So now if we are given points P_0 & P we can simplify further

Let

$x - x_0 = x$

$y - y_0 = y$

$z - z_0 = z$

Hence

$0 = ax + by + cz$

Generalizing, this equation doesn't have to equal zero, let this equation equate to unknown value d

Hence

$d = ax + by + cz$

with

$d = a \cdot x_0 + b \cdot y_0 + c \cdot z_0$

where

$\vec{n} = \langle a, b, c \rangle$

Point $P_0 = (x_0, y_0, z_0)$

d) For graphing in Mathematica, use computer.

$\boxed{2}$ Let point $P\ (1,2,3)$

Plane $Pl_1 \rightarrow x - y + z = 100$

So we are trying to find a line in the direction of the normal vector, $\vec{n}$, to plane Pl_1

Recall, that we say vector $\vec{v}$ is in the same direction as vector $\vec{u}$ if we can state $\vec{v}$ as the scalar-vector product of $\vec{u}$

Hence,

$\vec{v} = \alpha \cdot \vec{u}$ where $\vec{u}, \vec{v} \in \mathbb{R}^n \ (n = 2,3)$

$\alpha \in \mathbb{R}$.

Let's find normal vector $\vec{n}$

Recall, from q. $\boxed{1}$ in In Class Problem, Lesson 6.

scalar equation for a plane $\rightarrow d = a \cdot x + b \cdot y + c \cdot z$

where

$\vec{n} = \langle a, b, c \rangle$

So, given $Pl_1 \rightarrow \underset{a}{\underline{1}}x \underset{b}{\underline{-1}}y + \underset{c}{\underline{1}}z = \underset{d}{\underline{100}}$

Hence

$\vec{n} = \langle 1, -1, 1 \rangle$

We also are given point $(1,2,3)$.

Now, recall symmetric equation for line in $\mathbb{R}^3$ (Refer to Lesson 5.7)

① $x = x_0 + a \cdot t$

② $y = y_0 + b \cdot t$

③ $z = z_0 + c \cdot t$

Now let's rewrite this in terms of t.

① $t = \dfrac{x - x_0}{a}$

② $t = \dfrac{y - y_0}{b}$

③ $t = \dfrac{z - z_0}{c}$

Now we have three equations $(1,2,3)$ equating to t.

Hence,

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

Let point $P(1,2,3)$ correspond to $P(x_0, y_0, z_0)$

Also

$\vec{n} = \langle a, b, c \rangle = \langle 1, -1, 1 \rangle$.

Now substitute these values to get our line, L

$$L \rightarrow \frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-3}{1}$$

Hence

$$L \rightarrow x-1 = -1(y-2) = z-3$$

$\boxed{3}$. Let Plane 1, Pl₁ $\rightarrow x + 2z = 1$

Plane 2, Pl₂ $\rightarrow x + y - z = 0$

Recall from Lesson 6.3

Scalar equation for planes

$a \cdot x + b \cdot y + c \cdot z = d$

For Pl₁

$1 \cdot x + 0 \cdot y + 2 \cdot z = 1$

a b c d.

For Pl 2

$1 \cdot x + 1 \cdot y + (-1) \cdot z) = 0$.

a b c d

Recall from q. $\boxed{1}$ In-Class Problems, Lesson 6.

normal vector $\vec{n} = \langle a, b, c \rangle$

Hence,

For Pl₁, normal vector $\vec{n_1}$

$\vec{n_1} = \langle 1, 0, 2 \rangle$

For Pl₂, normal vector $\vec{n_2}$

$\vec{n_2} = \langle 1, 1, -1 \rangle$

Now let's find the direction vector orthogonal to $\vec{n_1}$ & $\vec{n_2}$

perpendicular

Recall that a cross product produces a vector orthogonal to both input vectors

Let's find $\vec{n_1} \times \vec{n_2}$ using the determinant method.

$$\begin{array}{|c|c|c|c|c|} i & j & k & i & j \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 1 & -1 & 1 & 1 \end{array}$$

$- \quad - \quad - \qquad + \quad + \quad +$

$\vec{n_1} = i + 0j + 2k$

$\vec{n_2} = i + j - k$.

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$\vec{n_1} \times \vec{n_2} = 0i + 2j + k$

$-2i - j - 0k \; +$

$-2i + j + k$

$\vec{n_1} \times \vec{n_2} = \langle -2, 1, 1 \rangle.$

Now, let's find a point on Pl_1 & Pl_2.

$Pl_1 \rightarrow x + 2z = 1$ ①

$Pl_2 \rightarrow x + y - z = 0$ ②

Let $z = 0$

① $x + 2(0) = 1$

Hence $x = 1$

② $x + y - z = 0$

$1 + y - 0 = 0$

$1 + y = 0$

$y = -1$

So we have a point $P(1, -1, 0)$

Recall from q. $\boxed{2}$ In-Class Problems Lesson 6

$x(t) = x_0 + a \cdot t$

$y(t) = y_0 + b \cdot t$

$z(t) = z_0 + c \cdot t$

Let $\vec{n_1} \times \vec{n_2} = \langle a, b, c \rangle = \langle -2, 1, 1 \rangle.$

$P(x_0, y_0, z_0) = (1, -1, 0)$

Hence Line L

$x(t) = 1 - 2t$

$y(t) = -1 + t$

$z(t) = t$

$\boxed{4}$. Distance from origin $O(0,0,0)$ to the plane $2x+y-2z=6$

Recall general equation for a plane

$a \cdot x + b \cdot y + c \cdot z = d$

where normal vector $\vec{n} = \langle a, b, c \rangle$ where $\vec{n} \in \mathbb{R}^3$

Given $2 \cdot x + 1 \cdot y + (-2) \cdot z = 6$

So in our case

$\vec{n} = \langle 2, 1, -2 \rangle$

Recall from our answer to question $\boxed{1}$ In-Class Problems, Lesson 6

$d = a \cdot x + b \cdot y + c \cdot z$

$d = a \cdot x_0 + b \cdot y_0 + c \cdot z_0$

So we can find distance, D from point P to plane, Pl

$$D = \frac{|a \cdot x_0 + b \cdot y_0 + c \cdot z_0 + d|}{\sqrt{a^2+b^2+c^2}}$$

Let

$\vec{n} = \langle a, b, c \rangle = \langle 2, 1, -2 \rangle$

$O = (0,0,0) = (x_0, y_0, z_0)$

$$D = \frac{|2 \cdot 0 + 1 \cdot 0 + (-2) \cdot 0 + 6|}{\sqrt{2^2 + 1^2 + (-2)^2}}$$

$$= \frac{|0+0+0+6|}{\sqrt{4+1+4}}$$

$$= \frac{|6|}{\sqrt{9}}$$

$$= \frac{6}{3}$$

$D = 2$

5. Plane 1, Pl_1 : $x+y-z=0$
Pl_2: $x-y=0$
Pl_3: $x+y+z=0$
Pl_4: $-2x-2y+2z=5$
Pl_5: $x+y=0$
Recall from Lesson 6.5.
Definition: Parallel Planes → have normal vectors that are parallel
Recall → direction of plane is the span of its normal vector
So given plane 1, Pl_1 gives normal vector $\vec{n_1}$
Pl_2 gives $\vec{n_2}$
Pl_1 is parallel to Pl_2 iff. $\vec{n_1} = \alpha \cdot \vec{n_2}$ where $\alpha \in \mathbb{R}$
in the same direction.
Definition: Orthogonal (perpendicular) planes
2 planes are orthogonal if their normal vectors are orthogonal
So Pl_1 gives $\vec{n_1}$
Pl_2 gives $\vec{n_2}$
Pl_1 & Pl_2 are orthogonal iff. $\vec{n_1} \cdot \vec{n_2} = 0$
dot product as measure of parallelity
perpendicular
So now let's find normal vectors to these planes
Pl_a gives $\vec{n}_a$
$Pl_1 \rightarrow \vec{n_1} = \langle 1,1,-1 \rangle$
$Pl_2 \rightarrow \vec{n_2} = \langle 1,-1,0 \rangle$
$Pl_3 \rightarrow \vec{n_3} = \langle 1,1,1 \rangle$
$Pl_4 \rightarrow \vec{n_4} = \langle -2,-2,2 \rangle$
$Pl_5 \rightarrow \vec{n_5} = \langle 1,1,0 \rangle$
We can see that Pl_1 & Pl_4 are parallel because
$\vec{n_4} = \alpha \cdot \vec{n_1}$
$\langle -2,-2,2 \rangle = -2 \cdot \langle 1,1,-1 \rangle$
$= \langle -2,-2,2 \rangle$
We can see that Pl_2 & Pl_5 are orthogonal because
$\vec{n_2} \times \vec{n_5} = 0$

$$\langle 1,-1,0\rangle \cdot \langle 1,1,0\rangle = 1\cdot 1 + 1\cdot(-1) + 0\cdot 0$$
$$= 1 - 1 + 0$$
$$= 0$$

6. Challenge problem

$$L_1(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 1-t \end{bmatrix}$$

We can separate this

$$L_1(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Point on L_1 direction vector

Likewise

$$L_2(s) = \begin{bmatrix} x(s) \\ y(s) \\ z(s) \end{bmatrix} = \begin{bmatrix} 1-s \\ 2+s \\ -2s \end{bmatrix}$$

$$L_2(s) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

plane $M \rightarrow 10x - 2y + 3z = 0$

Lets find P, point of intersection L_1 & L_2.

Equate $L_1(t)$ to $L_2(s)$

$t = 1 - s$

① $t + s = 1$

$2t = 2 + s$

② $2t - s = 2$

$1 - t = -2s$

③ $2s - t = -1$

using equation ① & ②

$$\begin{aligned} t + s &= 1 \\ 2t - s &= 2 \quad + \\ \hline 3t + 0 &= 3 \end{aligned}$$

$t = 1$

② $2t - s = 2$.

$2(1) - s = 2$

$2 - s = 2$

$-s = 0$

$s = 0$

Plugging in $t=1$ to $L_1(t)$

$P = (t, 2t, 1-t)$

$= (1, 2, 0)$

Let's check with $s=0$ to $L_2(s)$

$P = (1-s, 2+s, -2s)$

$= (1, 2, 0)$

Let's find Q, point of intersection L_1 & M

$M \rightarrow 10x - 2y + 3z = 0$

$L_1(t) = \langle t, 2t, 1-t \rangle$

$Q \rightarrow 10t - 2(2t) + 3(1-t) = 0$

$10t - 4t + 3 - 3t = 0$

$3t + 3 = 0$

$3t = -3$

$t = -1$

L_1 when $t = -1$

$Q = \langle -1, 2(-1), 1-(-1) \rangle$

$= \langle -1, -2, 2 \rangle$

Likewise, let's find R, point of intersection L_2 & M

$R \rightarrow 10(1-s) - 2(2+s) + 3(-2s) = 0$

$10 - 10s - 4 - 2s - 6s = 0$

$-18s + 6 = 0$

$6 = 18s$

$\frac{6}{18} = s$

$s = \frac{2}{3}$

$R = \left\langle 1 - \frac{2}{3}, 2 + \frac{2}{3}, -2\left(\frac{2}{3}\right) \right\rangle$

$R = \left(\frac{1}{3}, \frac{8}{3}, \frac{-4}{3}\right)$

Recall $A_{\text{triangle}} = \frac{1}{2} \cdot \text{base} \cdot \text{height}$

In our case

$\text{base} \cdot \text{height} = \|\overrightarrow{PQ} \times \overrightarrow{PR}\|_2$

$\overrightarrow{PQ} = Q - P$

$= (-1, -2, 2) - (1, 2, 0)$

$= (-2, -4, 2)$

$\overrightarrow{PR} = R - P$

$= \left(\frac{1}{3}, \frac{8}{3}, \frac{-4}{3}\right) - (1, 2, 0)$

$= \left(\frac{-2}{3}, \frac{2}{3}, \frac{-4}{3}\right)$

PQ x PR.

$$\begin{array}{c|c|c|c|c}
i & j & k & i & j \\
-2 & -4 & 2 & -2 & -4 \\
\frac{-2}{3} & \frac{2}{3} & \frac{-4}{3} & \frac{-2}{3} & \frac{2}{3}
\end{array}$$

$- \quad - \quad - \qquad\qquad + \quad + \quad +$

$\frac{16}{3} i - \frac{4}{3} j - \frac{4}{3} k$

$\frac{-4}{3} i - \frac{8}{3} j - \frac{8}{3} k \quad +$

$\frac{12}{3} i - \frac{12}{3} j - \frac{12}{3} k$

$PQ \times PR = \langle 4, -4, -4 \rangle$

$\|PQ \times PR\|_2 = \sqrt{4^2 + (-4)^2 + (-4)^2}$

$= \sqrt{16 + 16 + 16}$.

$= \sqrt{48}$

So A triangle $= \frac{1}{2} \| PQ \times PR \|_2$

$= \frac{1}{2} \cdot \sqrt{48}$

$= \frac{\sqrt{48}}{2}$

$= \frac{2\sqrt{12}}{2}$

$= \sqrt{12}$

$= 2\sqrt{3}$