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MIC, Lesson 5: In-Class Problems

1. Consider the following 2 functions → vector-valued

$r_2(t) = \langle x(t), y(t) \rangle$ and $r_3(t) = \langle x(t), y(t), z(t) \rangle$

$r_2(t) \in \mathbb{R}^2$ $\qquad$ $r_3(t) \in \mathbb{R}^3$

a) Draw a function map diagram in the form $f: D \to C$
Identify domain space & codomain.

For $r_2(t)$

$$\boxed{\vec{r_2}(t): \underbrace{D}_{\text{Domain}} \to \underbrace{\mathbb{R}^2}_{\text{Codomain}}}$$

Domain, $D \in \mathbb{R}$ we expect single, real number input (t) in function $\vec{r_2}(t)$
Codomain, $\mathbb{R}^2$ each output value of $\vec{r_2}(t)$ will be 2 vectors

For $r_3(t)$

$$\boxed{\vec{r_3}(t): D \to \mathbb{R}^3}$$

Domain, $D \in \mathbb{R}$, single, real number (t) input in function $\vec{r_3}(t)$
Codomain, $\mathbb{R}^3$, each output value of $\vec{r_3}(t)$ will be 3 vectors

b) Functions $r_2(t)$ and $r_3(t)$ are known as vector-valued functions because they are functions whose output is in the form of a vector. (Lesson 5.1)

$$t \to r_2(t) \to \left.\begin{matrix} (x(t)) \\ (y(t)) \end{matrix}\right\} \langle x(t), y(t) \rangle$$

$r_2(t)$ outputs 2 vectors

$$t \to r_3(t) \to \left.\begin{matrix} (x(t)) \\ (y(t)) \\ (z(t)) \end{matrix}\right\} \langle x(t), y(t), z(t) \rangle$$

$r_3(t)$ outputs 3 vectors

→ still in progress

c) Multi-variable real-valued functions

$$F: \underbrace{D}_{\text{Domain}} \to \underbrace{\mathbb{R}}_{\text{Codomain}}$$

$\mathbb{R}^2, \mathbb{R}^3$
2D, 3D

Domain, $D \in \mathbb{R}^n$ where $n = 2, 3$ we expect n real numbers to be input to function, F

Codomain, $\mathbb{R}$: each output of function F will be an individual real number.

single

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2.

$$L_1(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1+t \\ 2t \\ -1+3t \end{bmatrix} \qquad L_2(s) = \begin{bmatrix} 3+2s \\ 1+s \\ -2-s \end{bmatrix}$$

Find the point of intersection (x, y, z)

Like in $\mathbb{R}^2$ (2D) we can equate $L_1(t) = L_2(s)$

$L_1(t)$

$x(t) = 1+t$

$y(t) = 2t$

$z(t) = -1+3t$

$L_2(s)$

$x(s) = 3+2s$

$y(s) = 1+s$

$z(s) = -2-s$

$1+t = 3+2s$

$t-2s = 2$ **(a)**

$2t = 1+s$

$2t-s = 1$ **(b)**

$-1+3t = -2-s$

$3t+s = -1$ **(c)**

Let's solve for t, by using the system of equation b & c

$$\begin{array}{r} 2t - s = 1 \\ 3t + s = -1 \quad + \\ \hline 5t \quad = 0 \end{array}$$

$t = 0$

Let's solve for s using equation c

$3t+s = -1$

$3(0)+s = -1$

$s = -1$

Plugging these values into $L_1(t)$ or $L_2(s)$ will give us point of intersection

Given $L_1(t) = \begin{bmatrix} 1+t \\ 2t \\ -1+3t \end{bmatrix}$

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$$L_1(0) = \begin{bmatrix} 1+0 \\ 2(0) \\ -1+3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Let's check with $L_2(s)$

$$L_2(-1) = \begin{bmatrix} 3+2(-1) \\ 1+(-1) \\ -2-(-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Point of intersection $= \boxed{\langle 1, 0, -1 \rangle}$.

For the angle between $L_1(t)$ & $L_2(s)$

$$L_1(t) = \begin{bmatrix} 1+t \\ 2t \\ -1+3t \end{bmatrix} \qquad L_2(s) = \begin{bmatrix} 3+2s \\ 1+s \\ -2-s \end{bmatrix}$$

We can separate this into

$$L_1(t) = \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\text{point on } L_1} + t \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{\text{direction vector } \vec{a}} \qquad L_2(s) = \underbrace{\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}}_{\text{point on } L_2} + s \underbrace{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}}_{\text{direction vector } \vec{b}}$$

Recall

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|_2 \cdot \|\vec{b}\|_2 \cdot \cos\theta$$

Let's find $\vec{a} \cdot \vec{b}$

$$\vec{a} \cdot \vec{b} = \langle 1, 2, 3 \rangle \cdot \langle 2, 1, -1 \rangle$$
$$= 1 \cdot 2 + 2 \cdot 1 + 3 \cdot (-1)$$
$$= 1$$

$$\|\vec{a}\|_2^2 = 1^2 + 2^2 + 3^2$$
$$= 1 + 4 + 9$$
$$= 14$$
$$\|\vec{a}\|_2 = \sqrt{14}$$

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$\|\vec{b}\|_2^2 = 2^2+1^2+(-1)^2$

$= 4+1+1$

$= 6$

$\|\vec{b}\|_2 = \sqrt{6}$

$1 = \sqrt{14} \cdot \sqrt{6} \cdot \cos(\theta)$

$\cos(\theta) = \dfrac{1}{\sqrt{14} \cdot \sqrt{6}}$

$\theta = \cancel{\cos}^{-1} \left(\dfrac{1}{\sqrt{14} \cdot \sqrt{6}} \right)$

$\theta \approx \boxed{1.461}$

3. Find the equation for the plane that contains the curve

$r(t) = ti + \frac{1}{2} t^2 k$

I'm not sure how to solve this yet 18:12

4. $P(0,0,0)$
$Q(8,7,2)$

Recall, vector equation for line in $\mathbb{R}^3$

$\vec{r}(t) = \underbrace{\vec{r_0}}_{\text{point}} + t \cdot \underbrace{\vec{v}}_{\text{direction}}$

Let $\vec{r_0} = \langle 0,0,0 \rangle$

$\vec{v} = \cancel{PQ}\ \vec{PQ}$

$= \langle 8-0, 7-0, 2-0 \rangle$

$= \langle 8,7,2 \rangle$

$\vec{r}(t) = \langle 0,0,0 \rangle + t \langle 8,7,2 \rangle$

$= \boxed{\langle 8t, 7t, 2t \rangle}$

Now, let's restrict the domain to find the equation for the line segment between these 2 points

Let $\vec{r_0} = \langle 0,0,0 \rangle$

and

$\cancel{\vec{v}}\ \vec{r_1} = \langle 8,7,2 \rangle$

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$\vec{v} = \vec{r_1} - \vec{r_0}$

$= \langle 8,7,2\rangle - \langle 0,0,0\rangle$

$= \langle 8-0, 7-0, 2-0\rangle$

$= \langle 8,7,2\rangle$

Recall,

$\vec{r}(t) = \vec{r_0} + t\cdot\vec{v}$

$= \vec{r_0} + t\cdot(\vec{r_1} - \vec{r_0})$

$= \vec{r_0} + t\cdot\vec{r_1} + (-t\cdot\vec{r_0})$

$= \vec{r_0} - t\cdot\vec{r_0} + t\cdot\vec{r_1}$

$= \vec{r_0}\,(1-t) + t\cdot\vec{r_1}$

$\vec{r}t = (1-t)\,\vec{r_0} + t\cdot\vec{r_1}$

$= (1-t)\cdot\langle 0,0,0\rangle + t\cdot\langle 8,7,2\rangle$

$= \langle 0+0+0\rangle + \langle 8t, 7t, 2t\rangle$

$= \boxed{\langle 8t, 7t, 2t\rangle}$

5. Determine equation of line $r(\gamma)$ that is perpendicular to $L_1(t) = \langle 7t, 1+3t, 4t\rangle$ & $L_2(s) = \langle 1+s, -14+3s, -20+4s\rangle$ and passes through the point of intersection b/w $L_1(t)$ & $L_2(s)$

Let's find the point of intersection, i, by equating the 2 lines

$7t = 1+s$

$7t - s = 1 \quad (a)$

$1+3t = -14+3s$

$3t - 3s = -15 \quad (b_1)$

$t - s = -5 \quad (b_2)$

$4t = -20+4s$

$4t - 4s = -20 \quad (c_1)$

$t - s = -5 \quad (b_2)$

Using equation a & b_2, let's solve for t

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$$\begin{array}{l} 7t - \cancel{s} = 1 \\ t - \cancel{s} = -5 \quad - \\ \hline 6t \quad = -4 \end{array}$$

$$t = \frac{-4}{6}$$

$$t = \frac{-2}{3}$$

solve for s using equation b2

$$t - s = -5$$

$$\frac{-2}{3} - s = -5$$

$$-s = -5 + \frac{2}{3}$$

$$-s = \frac{-13}{3}$$

$$s = \frac{13}{3}$$

Using these values for s & t plug into L1(t)

$$L1(t) = 7\cdot\left(\frac{-2}{3}\right) = \frac{-14}{3}$$

$$= 1 + 3\left(\frac{-2}{3}\right) = -1$$

$$4\left(\frac{-2}{3}\right) = \frac{-8}{3}$$

Point of intersection $i = \left\langle \frac{-14}{3}, -1, \frac{-8}{3} \right\rangle$

Now, to find equation of the line
Recall, that in $\mathbb{R}^3$

$$r(\gamma) = \vec{r_0} + \gamma \cdot \vec{v}$$

point on $r(\gamma)$

direction of $r(\gamma)$

Let $i = \vec{r_0}$

$$= \left\langle \frac{-14}{3}, -1, \frac{-8}{3} \right\rangle$$

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$L_1(t) = \langle 7t, 1+3t, 4t \rangle$

$= \underbrace{\langle 0,1,0 \rangle}_{\text{point on } L_1} + t \underbrace{\langle 7,3,4 \rangle}_{\to \text{ direction vector } L_1,\ \vec{a}}$

$L_2(s) = \langle 1+s, -14+3s, -20+4s \rangle$

$= \underbrace{\langle 1,-14,-20 \rangle}_{\text{point on } L_2} + s \underbrace{\langle 1,3,4 \rangle}_{\to \text{ direction vector } L_2,\ \vec{b}}$

Do we use cross product to find $(\vec{a} \times \vec{b})$?

Recall

$\vec{a} \times \vec{b} = \langle y_1 z_2 - z_1 y_2, x_2 z_1 - x_1 z_2, x_1 y_2 - x_2 y_1 \rangle$

$= \langle \overset{x_1}{7}, \overset{y_1}{3}, \overset{z_1}{4} \rangle \times \langle \overset{x_2}{1}, \overset{y_2}{3}, \overset{z_2}{4} \rangle$

$= \langle 3 \cdot 4 - 4 \cdot 3, 1 \cdot 4 - 7 \cdot 4, 7 \cdot 3 - 1 \cdot 3 \rangle$

$= \langle 12-12, 4-28, 21-3 \rangle$

$\vec{a} \times \vec{b} = \langle 0, -24, 18 \rangle$

Now, lets plug our values into vector equation in $\mathbb{R}^3$

$r(\gamma) = \left\langle \frac{-14}{3}, -1, \frac{-8}{3} \right\rangle + \gamma \langle 0, -24, 18 \rangle$

$= \boxed{\left\langle \frac{-14}{3}, -1-24\gamma, \frac{-8}{3} + 18\gamma \right\rangle}$

6. $y = -5$

$r(t) = \langle \underset{x}{4t+1}, \underset{y}{-1+4t}, \underset{z}{t-6} \rangle$

we can rewrite this line as

$r(t) = \langle 1, -1, -6 \rangle + t \langle 4, 4, 1 \rangle$

As $y = -5$

Let's substitute $y = -1+4t$

$-1+4t = -5$

$4t = -4$

$t = -4$

point of intersection $= \langle 4(-4)+1, -1+4(-4), (-4)-6 \rangle$

$\boxed{= (-15, -17, 24)}$

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7. Circle in $\mathbb{R}^2$

$$\underbrace{(x-h)^2}_{a^2} + \underbrace{(y-k)^2}_{b^2} = \underbrace{r^2}_{c^2}$$

→ this is like Phytagoras

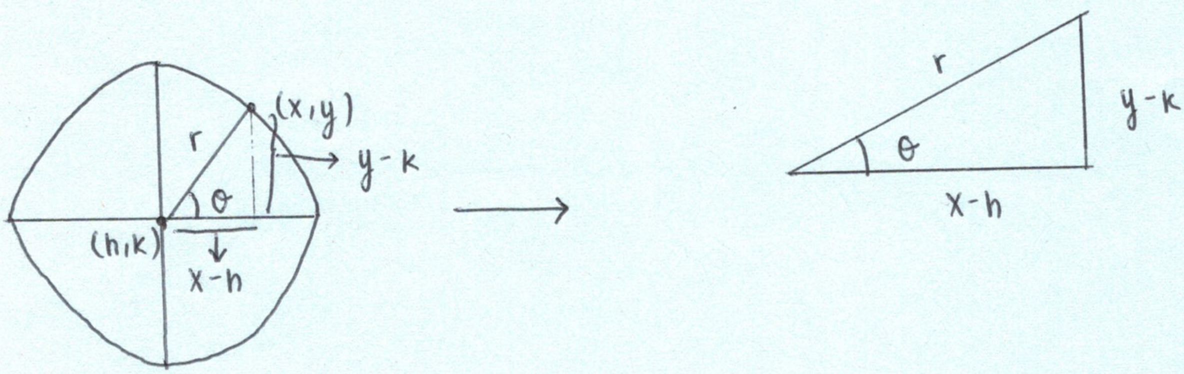
Let's rewrite this equation

$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$

Sketching a circle to illustrate this

With a circle centered at (h,k)

& radius, r



Using phytagorean theorem we see the equation for a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Recalling our trigonometric operators $\cos(\theta)$ & $\sin(\theta)$

$$\cos(\theta) = \frac{A}{H} = \frac{x}{r}$$

$x = r \cdot \cos(\theta)$ → note: $x(\theta) = r \cdot \cos(\theta)$

$$\sin(\theta) = \frac{O}{H} = \frac{y}{r}$$

function in terms of θ

$y = r \cdot \sin(\theta)$ → note: $y(\theta) = r \cdot \sin(\theta)$

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So now
let $\theta = t$

$y(\theta) = r \cdot \sin(\theta)$

$y(t) = r \cdot \sin(t)$

$x(\theta) = r \cdot \cos(\theta)$
$x(t) = r \cdot \cos(t)$

Recall, the basic format of our vector valued function
$r(t) = \langle x(t), y(t) \rangle$
we get
$$\boxed{r(t) = \langle r \cdot \cos(t), r \cdot \sin(t) \rangle}$$

8.

A. $\frac{x^2}{25} + \frac{y^2}{4} = 1$

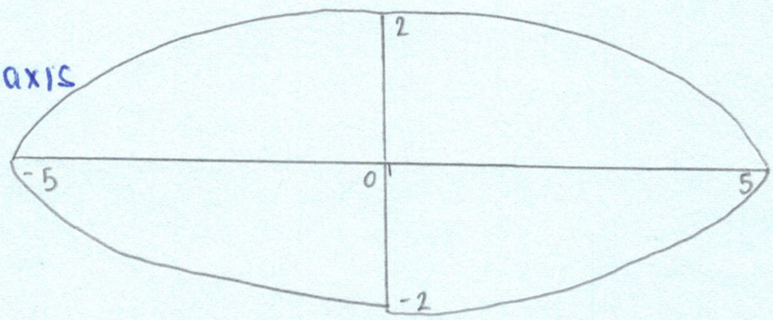
Recall from Lesson 5.6
Scalar equation of ellipses in $\mathbb{R}^2$
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

In this case
$\left.\begin{array}{l} h = 0 \\ k = 0 \end{array}\right\}$ this ellipses is centered at $(0,0)$

$\left.\begin{array}{l} a^2 = 25 \\ a = \pm 5 \end{array}\right\}$ Recall a = x-semiaxis

$\left.\begin{array}{l} b^2 = 4 \\ b = \pm 2 \end{array}\right\}$ b = y-semiaxis

Sketching this



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B. $\dfrac{x^2}{9} + \dfrac{y^2}{36} = 1$

By the same process as A.

$\left.\begin{aligned} h &= 0 \\ k &= 0 \end{aligned}\right\} (h,k) = (0,0)$

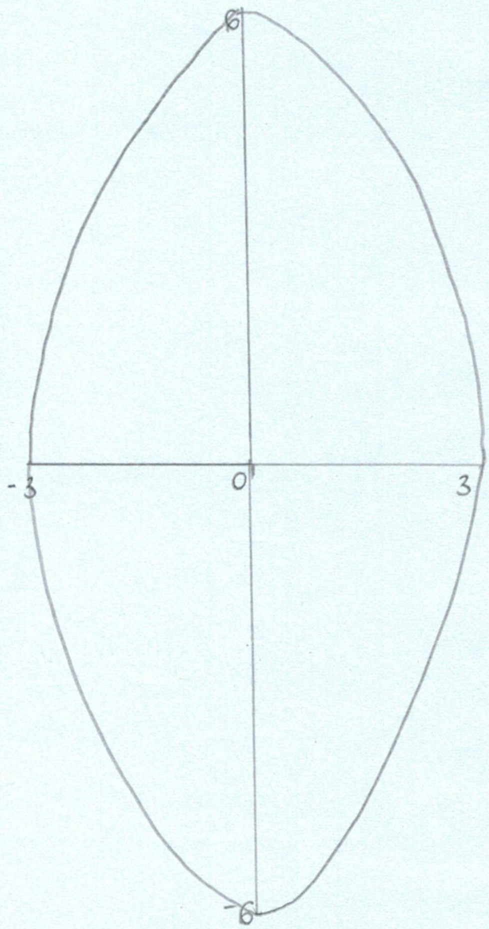
$a^2 = 9$

$a = \pm 3$

$b^2 = 36$

$b = \pm 6$

sketching this we get



C. $12x^2 + 5y^2 = 60$

$60 = 5 \times 12$

so rewriting this in the form

$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

$= \dfrac{12x^2}{60} + \dfrac{5y^2}{60} = 1$

$= \dfrac{x^2}{5} + \dfrac{y^2}{12} = 1$

$\left.\begin{aligned} h &= 0 \\ k &= 0 \end{aligned}\right\} (h,k) = (0,0) \rightarrow$ ellipses centered at $(0,0)$

$a^2 = 5$

$a = \pm\sqrt{5}$

$b^2 = 12$

$b = \pm\sqrt{12}$

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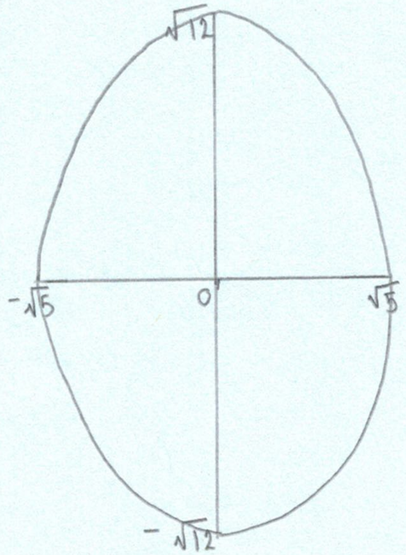
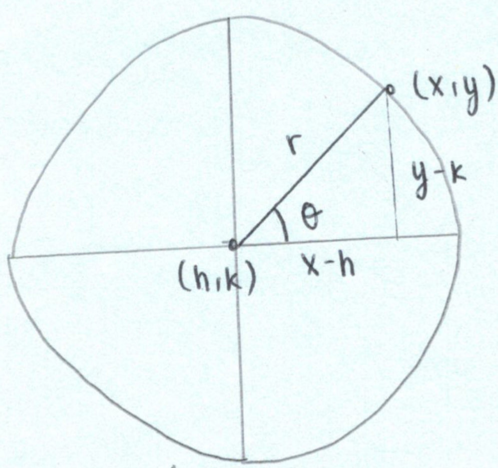
Sketching this



9. Recall, equation for ellipse in $\mathbb{R}^2$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

An ellipse is the generalization of a circle.
Let's sketch a circle centered at (h,k) with radius r. Circle, C_1



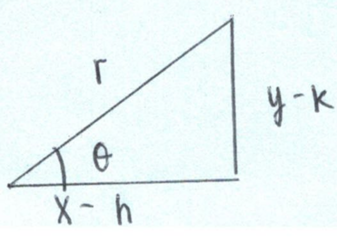
Recall lesson 5.4

given a circle, C_1 ~~cent~~

$C_1 \{(x,y) : \sqrt{(x-h)^2 + (j-k)^2 = r^2}\}$

we can derive this from phytagoras

↓



using phytagoras

$c^2 = a^2 + b^2$

let $c = r$

$a = (x-h)$

$b = (y-k)$

so $(x-h)^2 + (y-k)^2 = r^2$

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We can rewrite this as

$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$

Now we recall that in lesson 2.8 we define a circle, C, as the collection of points (x,y) equidistant to the center (h,k) by the distance of the radius, r.

↳ same distance

Now let's generalize this for an ellipse

Let

$r = a$ for the x-semi-axis
where $a \in \mathbb{R}$

&

$r = b$ for the y-semiaxis
where $b \in \mathbb{R}$

rewriting this we get

$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$$ → equation for ellipse in $\mathbb{R}^2$

Returning to question 3 → still in progress

3. $r(t) = t\,i + \frac{1}{2}t^2 K$

Let $i = x$
$K = z$

$r(t) = xt + z\frac{1}{2}t^2$

So breaking $r(t)$ as a vector-valued function where

$x(t) = t$
$y(t) = 0$
$z(t) = \frac{1}{2}t^2$

To write the equation of a plane, we need:

① A point

② Normal vector.