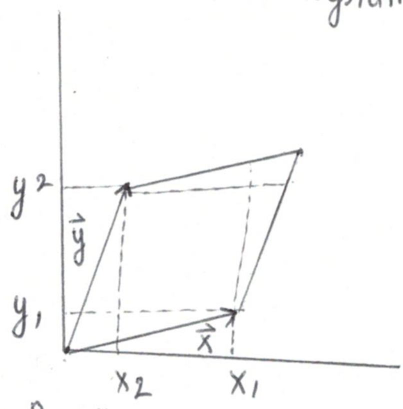
Jeff's Notes

1. $\vec{x} = \langle x_1, y_1 \rangle$
$\vec{y} = \langle x_2, y_2 \rangle$ where $\vec{x}$ & $\vec{y} \in \mathbb{R}^2$

a) Find area of parallelogram



~~Are~~ Recall formula for area of parallelogram from Lesson 4.3

area of parallelogram = $\boxed{y_2 \cdot x_1 - x_2 \cdot y_1}$

b) Taking the above formula.

$A_{\text{parallelogram}} = y_2 \cdot x_1 - x_2 \cdot y_1$

for 2 input vectors, $\vec{x}$ & $\vec{y}$ in which $\vec{x}$ & $\vec{y} \in \mathbb{R}^2$

To generalize this for vectors in $\mathbb{R}^3$, let us now redefine

$\vec{x} = \langle x_1, y_1, z_1 \rangle$

$\vec{y} = \langle x_2, y_2, z_2 \rangle$

For the parallelogram in the xy-plane

$A_{1,2} = x_1 \cdot y_2 - x_2 \cdot y_1$

For the parallelogram in the xz-plane

$A_{1,3} = x_1 \cdot z_2 - x_2 \cdot z_1$

☐ This is great work! You are clearly spending lots of time on these solutions (which you can track because you're writing time stamps on each page !! 🙂)

☐ I love and respect this work ethic!

If you can maintain and refine this over many years, I have a feeling you will realize your dreams. Please continue to develop & improve your work ethic and study systems. This takes lots of conscious meditation & hard work.

☐ I made some notes on how you might improve your solutions (from my perspective)! Enjoy!

For the parallelogram in the yz-plane

$A_{2,3} = y_1 \cdot z_2 - y_2 \cdot z_1$

From this, we can come up with the cross product of

$\vec{x} \times \vec{y}$

Recall

$\vec{i} = \langle 1, 0, 0 \rangle$

$\vec{j} = \langle 0, 1, 0 \rangle$

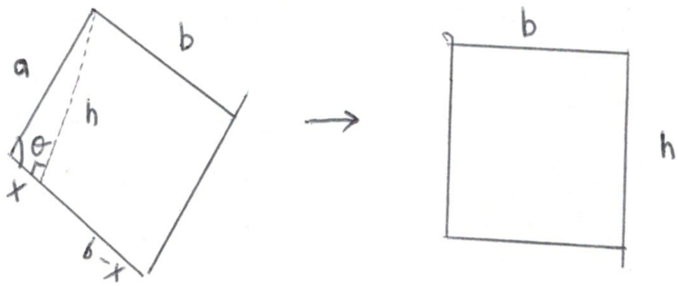
$\vec{k} = \langle 0, 0, 1 \rangle$

$$\rightarrow \vec{x} \times \vec{y} = (y_1 \cdot z_2 - y_2 \cdot z_1) \cdot \vec{i} + (x_1 \cdot z_2 - x_2 \cdot z_1) \cdot -\vec{j} + (x_1 \cdot y_2 - x_2 \cdot y_1) \cdot \vec{k}$$

$$= (y_1 \cdot z_2 - y_2 \cdot z_1) \cdot \vec{i} + (x_2 \cdot z_1 - x_1 z_2) \cdot \vec{j} + (x_1 \cdot y_2 - x_2 \cdot y_1) \cdot \vec{k}$$

$$\boxed{\vec{x} \times \vec{y} = \langle y_1 z_2 - y_2 z_1, x_2 z_1 - x_1 z_2, x_1 y_2 - x_2 y_1 \rangle}$$

c)



A rectangle = $b \cdot h$

A parallelogram = $b \cdot h$

$= a \cdot b \cdot \sin\theta$

If $b = \vec{x}$

and $a = \vec{y}$

then

$A_{\text{parallelogram}} = \|\vec{x}\|_2 \cdot \|\vec{y}\|_2 \cdot \sin(\theta)$

θ angle between $\vec{x}$ & $\vec{y}$

So

$$\boxed{\|\vec{x} \times \vec{y}\|_2 = \|\vec{x}\|_2 \cdot \|\vec{y}\|_2 \cdot \sin(\theta)}$$

2. For any vectors $u, v \in \mathbb{R}^3$, $(u \times v) \cdot u = 0$ ← Jeff's suggestions

u & v
2 vectors in $\mathbb{R}^3$
(3-D)

☐ clearly delineate the problem statement and make it easy to see where your solution begins. You might even use a labeling system:

$\boxed{\text{Problem:}}$ Blah blah blah

Solution: Blah blah blah

$(u \times v) \cdot u = 0$

Recall lesson 3.9 which asserts that 2 non-zero vectors, $\vec{x}$ & $\vec{y}$ are orthogonal if & only if

$\vec{x} \cdot \vec{y} = 0$

perpendicular

Given,

$\underbrace{(u \times v)}_{\vec{x}} \cdot \underbrace{u}_{\vec{y}} = 0$

we are claiming:

vectors $(u \times v)$ & $u \in \mathbb{R}^3$

& vector $(u \times v)$ is orthogonal to one another

perpendicular.

Recall, Lesson 4.8 ← This is a great habit: specifically reference lecture content !!

Geometric property 2 → the output of the cross product is orthogonal to the input vectors.

Let's prove this,

Let $\vec{u} = \langle x_1, y_1, z_1 \rangle$

$\vec{v} = \langle x_2, y_2, z_2 \rangle$

Then, consider

$u \cdot (u \times v) = \langle x_1, y_1, z_1 \rangle \cdot (u \times v)$

$= x_1(y_2 z_1 - y_1 z_2)$
$+ y_1(x_2 z_1 - x_1 z_2)$
$+ z_1(x_2 y_1 - x_1 y_2)$

$= \cancel{x_1 y_2 z_1} - x_1 \cancel{y_1 z_2}$
$+ \cancel{y_1 x_2 z_1} - x_1 \cancel{z_2 y_1}$
$+ \cancel{z_1 x_2 y_1} - \cancel{z_1 x_1 y_2}$

$= 0$

so $(u \times v) \cdot u = 0$.
therefore $(u \times v) \cdot u$ is orthogonal to one another.

$\boxed{\text{True}}$.

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I would label page numbers & count for entire document

3. $u, v \in \mathbb{R}^3$

$(u-v) \times (u+v) = 2u \times v$

Let $u = \langle x_1, y_1, z_1 \rangle$
$v = \langle x_2, y_2, z_2 \rangle$

Now, let's find $u-v$

$u - v = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle$
$= \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$

Now, let's find $u+v$

$u + v = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle$
$= \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$

Having found $u-v$ & $u+v$
Let's find $(u-v) \times (u+v)$
$= \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle \times \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$

$$\begin{array}{ccc|cc} i & j & k & i & j \\ x_1 - x_2 & y_1 - y_2 & z_1 - z_2 & x_1 - x_2 & y_1 - y_2 \\ x_1 + x_2 & y_1 + y_2 & z_1 + z_2 & x_1 + x_2 & y_1 + y_2 \\ - & - & - & + & + \quad + \end{array}$$

this attempt at scalar notation is courageous but extremely tedius... Too much so in my opinion.

Note: I would avoid a scalar approach to this problem and instead work in vectors. See video Lesson 4.7: Algebraic Properties of cross product...

3. We can prove this in 2 ways, geometrically & algebraically.

Algebraically: Using vector notation

$(\vec{u}-\vec{v}) \times (\vec{u}+\vec{v}) = 2\vec{u} \times \vec{v}$

Using the distributive property of the cross product

$(u - v) \times (u+v)$ Left-hand side

$= u \times u + \cancel{u \times v} - \cancel{v \times u} - v \times v$

$= u \times u - v \times v$

$2\vec{u} \times \vec{v}$

$= 2\vec{u} \times 2\vec{v}$

$= \vec{u} \times \vec{u} - \vec{v} \times \vec{v}$ ← yes: this is it! You might use vector hats to be explicit (or harpoons $\rightharpoonup$)

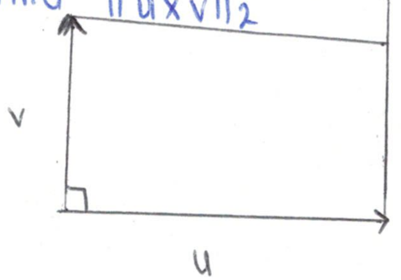
$\boxed{\text{True}}$

$(u-v) \times (u+v) = 2(u \times v)$

4. Suppose $u, v \in \mathbb{R}^3$ are orthogonal vectors
perpendicular

Find magnitude $(u \times v)$

Find $\|u \times v\|_2$



Recall lesson 4.9

$$\|u \times v\|_2 = \|u\|_2 \times \|v\|_2 \cdot \sin(\theta)$$
$$= \|u\|_2 \times \|v\|_2 \cdot \sin(90)$$
$$= \|u\|_2 \times \|v\|_2 \cdot 1$$
$$= \boxed{\|u\|_2 \cdot \|v\|_2}$$

5. Find a vector normal to $\langle 8,0,3\rangle$ & $\langle -7,1,2\rangle$

perpendicular

let

$x = \langle 8,0,3\rangle$

$y = \langle -7,1,2\rangle$ where $x, y \in \mathbb{R}^3$

Recall lesson 4.8

Geometric property 2: The output of the cross product is orthogonal to both input vectors

$\vec{x} \times \vec{y} \perp \vec{x}$

&

$\vec{x} \times \vec{y} \perp \vec{y}$

To find a vector normal to $\vec{x}$ & $\vec{y}$ we need to find $(\vec{x} \times \vec{y})$

Using the determinant method, let's find $(\vec{x} \times \vec{y})$

$$(\vec{x} \times \vec{y}) = \left| \begin{array}{ccc|cc} i & j & k & i & j \\ 8 & 0 & 3 & 8 & 0 \\ -7 & 1 & 2 & -7 & 1 \end{array} \right.$$

$$- \quad - \quad - \quad + \quad + \quad +$$

$= -3i + 0i - 16j - 21j - 0k + 8k$

$= -3i - 37j + 8k$

$(\vec{x} \times \vec{y}) = \boxed{\langle -3, -37, 8\rangle}$

6. The triangle vertices are located at $\underbrace{(0,0,0)}_{A}$, $\underbrace{(1,0,-1)}_{B}$ & $\underbrace{(1,-1,2)}_{C}$
Let origin, O, be the fixed point

Let's find the position vectors.

Make sure to write full problem statement

Let A be point (0,0,0)
B be point (1,0,-1)
C be point (1,-1,2)

Position vector A $= \vec{OA} = \langle 0,0,0 \rangle$
B $= \vec{OB} = \langle 1,0,-1 \rangle$
C $= \vec{OC} = \langle 1,-1,2 \rangle$

Let us now find the base & height of the triangle
Let $\vec{AB}$ = base

$$\vec{AB} = \vec{OB} - \vec{OA}$$
$$= \langle 1,0,-1 \rangle - \langle 0,0,0 \rangle$$
$$= \langle 1,0,-1 \rangle$$

Let $\vec{AC}$ = height

$$\vec{AC} = \vec{OC} - \vec{OA}$$
$$= \langle 1,-1,2 \rangle - \langle 0,0,0 \rangle$$
$$= \langle 1,-1,2 \rangle$$

Recall

A triangle $= \frac{1}{2} b \cdot h$
$$= \frac{1}{2} \cdot \vec{AB} \times \vec{AC}$$

Using cross product lets find AB x AC

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{vmatrix} \begin{matrix} i & j \\ 1 & 0 \\ 1 & -1 \end{matrix}$$

$$= 0i - i - j - 2j - k + 0k$$

$$= -i - 3j - k$$

$$\vec{AB} \times \vec{AC} = \langle -1, -3, -1 \rangle$$

Finding the two norm of this

$$\|AB \times AC\|_2^2 = (-1)^2 + (-3)^2 + (-1)^2$$

$$= 1 + 9 + 1$$

$$= 11$$

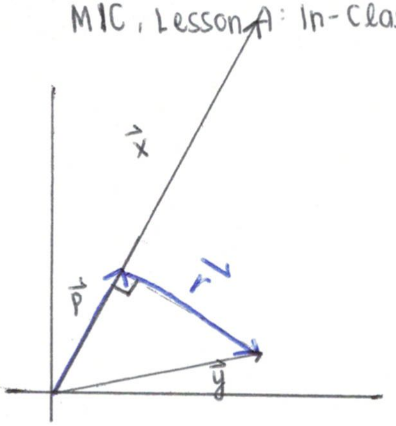
$$\|AB \times AC\|_2 = \sqrt{11}$$

$$A_{\text{triangle}} = \frac{1}{2} \cdot \|AB \times AC\|_2$$

$$= \frac{1}{2} \cdot \sqrt{11}$$

$$= \boxed{\frac{\sqrt{11}}{2}}$$

7.



Let us define $\vec{p}$ to be the projection of $\vec{y}$ in the direction of $\vec{x}$.

So

$$\vec{p} = \alpha \cdot \vec{x}$$

where $\alpha \in \mathbb{R}$ & is unknown.

To do this orthogonally / so that $\vec{p}$ is an orthogonal projection, we need to consider residual vector, $\vec{r}$

$$\vec{r} = \vec{y} - \vec{p}$$

$\rightarrow \vec{r} = \vec{y} - \alpha \cdot \vec{x}$

$\rightarrow \vec{x} \cdot \vec{r} = 0$ $\quad$ $\vec{x}$ & $\vec{r}$ are orthogonal to each other

$\rightarrow \vec{x} \cdot (\vec{y} - \alpha \cdot \vec{x}) = 0$

$\rightarrow \vec{x} \cdot \vec{y} - \alpha \cdot \vec{x} \cdot \vec{x} = 0$

$\rightarrow \alpha \cdot \vec{x} \cdot \vec{x} = \vec{x} \cdot \vec{y}$

$$\alpha = \frac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}}$$

Recall $\vec{x} \cdot \vec{x} = \|\vec{x}\|_2$

$$= \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|_2}$$

I like that you are repeating this derivation for yourself as you solve the problem. Can you do it from memory yet?

I used to set a goal for myself when doing homework: derive each formula I use in class from first principles at least 15 times over many different days... This is hard work but it pays off over time.

Returning back to our equation

$$\vec{p} = \alpha \cdot \vec{x}$$

$$= \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|_2} \cdot \vec{x}$$

Recall $\frac{\vec{x}}{\|\vec{x}\|_2}$ = unit vector of $\vec{x}$

$$= \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|_2} \cdot \frac{\vec{x}}{\|\vec{x}\|_2}$$