

Section 5.3 : Diagonalization

Similarity p. 277

Definition : Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$.

We say A is similar to B if there is an invertible matrix P s.t.

$$P^{-1} A P = B \quad \text{or equivalently}$$

$$P B P^{-1} = A$$

We denote this as $A \sim B$

Note:

Transforming A into $P^{-1}AP$

is called a similarity transformation

(or conjugation)

Theorem 4 Let $A, B \in \mathbb{R}^{n \times n}$.

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If $A \sim B$, then A and B have the same characteristic polynomials.

Proof: Suppose $A \sim B$.

We know there is an invertible P st. $B = P^{-1}AP$.

Consider

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P$$

$$= P^{-1}(A - \lambda I)P$$

$$\Rightarrow \det(B - \lambda I) = \det(P^{-1}(A - \lambda I)P)$$

$$= \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det(P)$$

$$= \det(A - \lambda I). \square$$

Example 3 : Diagonalizing Matrices

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Recall as our warm up problem on Wednesday

we found the eigen vector, eigenvalue pairs
of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

We saw

$$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4$$

$$= -(\lambda-1)(\lambda+2)^2$$

$$\Rightarrow \lambda_1 = 1, \underbrace{\lambda_2 = \lambda_3 = -2}$$

algebraic multiplicity 2

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} : A\vec{v}_1 = \lambda_1 \vec{v}_1,$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} : A\vec{v}_2 = \lambda_2 \vec{v}_2,$$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} : A\vec{v}_3 = \lambda_3 \vec{v}_3$$

3 linearly independent
eigen vectors \Rightarrow A diagonalizable
by thm 5

Thus we see if $P = [\vec{v}_1 \vec{v}_2 \vec{v}_3]$

$$AP = A[\vec{v}_1 \vec{v}_2 \vec{v}_3] = [A\vec{v}_1 A\vec{v}_2 A\vec{v}_3]$$

$$= \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -2 \\ -1 & +2 & 0 \\ 1 & 0 & +2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= P D$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}}_A = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}}_P \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}}_D \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{bmatrix}}_{P^{-1}}$$

Algorithm to Diagonalize a Matrix

Let $A \in \mathbb{R}^{n \times n}$

Step 1: Find eigenvalues & eigenvectors of A

→ Step 2: Check if there are n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. If yes, go to step 3, else stop.

Step 3: Construct matrix $P = [\vec{v}_1 | \dots | \vec{v}_n]$

Step 4: Construct $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ where $A\vec{v}_k = \lambda_k \vec{v}_k$

This is an important step. Not all matrices are diagonalizable

Example 4 : Diagonalize the matrix

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$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Solution: We will follow our algorithm.

Theorem
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If $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$ of an $n \times n$ matrix A , then the set $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_r\}$ is linearly independent.

Proof: Suppose, hoping for contradiction, that $\{\vec{v}_1, \dots, \vec{v}_r\}$ forms a linearly dependent set.

Then there exists a $p \in \mathbb{N}$ ($1 \leq p < r$) such that

$$\vec{v}_{p+1} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_p \vec{v}_p$$

$$\Rightarrow A \vec{v}_{p+1} = A(c_1 \vec{v}_1 + \dots + c_p \vec{v}_p)$$

$$= c_1 A \vec{v}_1 + \dots + c_p A \vec{v}_p$$

$$= c_1 \lambda_1 \vec{v}_1 + \dots + c_p \lambda_p \vec{v}_p$$

$$= \lambda_{p+1} \vec{v}_{p+1}$$

$$\Rightarrow c_1 (\lambda_1 - \lambda_{p+1}) \vec{v}_1 + \dots + c_p (\lambda_p - \lambda_{p+1}) \vec{v}_p = \vec{0}$$

$$\Rightarrow c_i (\lambda_i - \lambda_{p+1}) = 0 \quad \Rightarrow \quad c_i = 0 \quad \Rightarrow \quad \vec{v}_{p+1} = \vec{0} \Rightarrow \text{E}$$

Theorem 5
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An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Proof: \Rightarrow Suppose A is diagonalizable.
(only if)

Then there exists an invertible $P \in \mathbb{R}^{n \times n}$ and a diagonal $D \in \mathbb{R}^{n \times n}$ such that

$$A = P D P^{-1}$$

$$\Rightarrow AP = PD$$

$$\Rightarrow A[\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n] = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]D$$

$$\Rightarrow [A\vec{v}_1 \ | A\vec{v}_2 \ | \dots \ | A\vec{v}_n] = [\lambda_1\vec{v}_1 \ | \lambda_2\vec{v}_2 \ | \dots \ | \lambda_n\vec{v}_n]$$

$$\Rightarrow A\vec{v}_k = \lambda_k \vec{v}_k \quad \text{for } k=1,2,\dots,n$$

$\Rightarrow \lambda_k$ is an eigenvalue of A and \vec{v}_k is the eigenvector corresponding to eigenvalue λ_k .

Further, since P is invertible and $P(:, k) = \vec{v}_k$, we know $\{\vec{v}_k\}_{k=1}^n$ is linearly independent. \blacksquare

\Leftarrow
(if)

Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent eigenvectors of A , where $A\vec{v}_k = \lambda_k \vec{v}_k$ for associated eigenvalue λ_k .

Let $P = [\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_n]$.

Since $\{\vec{v}_k\}_{k=1}^n$ linearly independent, $\det(P) \neq 0$.

Let $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$.

Then $AP = PD$

$$\Rightarrow A = P D P^{-1}$$

$\Rightarrow \exists P$ invertible s.t. $A \sim D$ diagonal

$\Rightarrow A$ diagonalizable. \square

Thus we've shown both directions.

Theorem 5 Notes:

□ Another way to state Thm 5 is

$A \in \mathbb{R}^{n \times n}$ diagonalizable if and only if

~~there is a basis~~ the set of all linearly independent eigenvectors of A form a basis of \mathbb{R}^n .

□ $A \in \mathbb{R}^{n \times n}$ diagonalizable $\Leftrightarrow \exists P$ invertible and D diagonal with $A = PDP^{-1}$

- Columns of P are the n linearly independent eigenvectors of A
- Diagonal entries of D are the eigenvalues of A that correspond to eigenvectors in P (in same order).

Theorem 5 : Prerequisites

Let $A \in \mathbb{R}^{n \times n}$ be given.

Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^n$ s.t. $P = [\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_n] \in \mathbb{R}^{n \times n}$,

where $P(:, k) = \vec{v}_k$ for $k = 1, 2, \dots, n$.

Suppose $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ and $D \in \mathbb{R}^{n \times n}$ is defined by

$$D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

$$= \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix}_{n \times n}$$

Note:

□ Here $d_{ii} = \lambda_i$ for $i = 1, 2, \dots, n$

and $d_{ik} = 0$ if $i \neq k$

$$\text{Then } AP = A [\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_n]$$

$$= [A\vec{v}_1 | A\vec{v}_2 | \dots | A\vec{v}_n] \quad \blacksquare$$

We also notice that

$$PD = [\tilde{v}_1 \mid \tilde{v}_2 \mid \cdots \mid \tilde{v}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$= [\lambda_1 \tilde{v}_1 \mid \lambda_2 \tilde{v}_2 \mid \cdots \mid \lambda_n \tilde{v}_n]_{nxn} \quad \boxed{\text{II}}$$

These two properties are verified using the definition of matrix-matrix multiplication.