

Section 5.2: The Characteristic Equation

Warm Up: Find the eigenvalues and eigenvectors of the matrix

Example 3:

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$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

Solution: To find the eigenvector/eigenvalue pairs, we begin as before

$$0 = \det(A - \lambda I)$$

$$= \det \left(\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

Note:

Since $\det(A - \lambda I)$ is calculated using 6 terms, we have the ability to rely on our technique.

$\boxed{1} (1-\lambda)(-5-\lambda)(1-\lambda) = -(\lambda^2 - 2\lambda + 1)(\lambda + 5)$
 $= -(\lambda^3 + 3\lambda^2 - 9\lambda + 5)$
 $= -\lambda^3 - 3\lambda^2 + 9\lambda - 5$

$\boxed{2} -3 \cdot 3^2 = -27$

$\boxed{3} 3^2 \cdot -3 = -27$

$\boxed{4} +9(\lambda+5) = 9\lambda + 45$

$\boxed{5} -9(\lambda-1) = -9\lambda + 9$

$\boxed{6} -9(\lambda-1) = -9\lambda + 9$

$$\Rightarrow \det(A - \lambda I) = (-\lambda^3 - 3\lambda^2 + 9\lambda - 5) - 9\lambda + 9$$

$$= -\lambda^3 - 3\lambda^2 + 4 = p(\lambda)$$

Note:

Since this characteristic polynomial must be factorable in some form

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = -\lambda^3 - 3\lambda^2 + 4$$

$$\Rightarrow -\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 4$$

$$\Rightarrow \lambda_i = \pm 1, \pm 2, \pm 4$$

We can check $p(\lambda)$ at these possible roots. We note $p(1) = -1^3 - 3 \cdot (1)^2 + 4 = 0$

$\Rightarrow (\lambda - 1)$ is a factor of $p(\lambda)$

\Rightarrow We can factor $\lambda - 1$ out of $p(\lambda)$ using polynomial long division

$$\begin{array}{r} -\lambda^2 - 4\lambda + -4 \\ \lambda - 1 \quad \boxed{-\lambda^3 - 3\lambda^2 + 0\lambda + 4} \\ - [-\lambda^3 + \lambda^2] \\ \hline -4\lambda^2 + 0\lambda \\ - [-4\lambda^2 + 4\lambda] \\ \hline -4\lambda + 4 \\ - [-4\lambda + 4] \\ \hline 0 \end{array}$$

O remainder

$$\Rightarrow p(\lambda) = -\lambda^3 - 3\lambda^2 + 4$$

$$= -(\lambda - 1)(\lambda^2 + 4\lambda + 4)$$

$$= -(\lambda - 1)(\lambda + 2)^2$$

Since we know the roots of the characteristic polynomial are given $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = -2$, these are the eigenvalues of matrix A .

To find the corresponding eigenvectors, we want to find the basis vectors for $\text{Null}(A - \lambda I)$ for each value of λ .

Consider $\lambda_1 = 1$:

$$(A - \lambda_1 I) \vec{x}_1 = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda_1 & 3 & 3 \\ -3 & -5-\lambda_1 & -3 \\ 3 & 3 & 1-\lambda_1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \vec{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \text{ is a basis for eigenspace } E_{\lambda_1}.$$

Consider $\lambda_2 = \lambda_3 = -2$:

$$(A - \lambda_2 I) \vec{x}_2 = \vec{0}$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \boxed{\vec{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \quad \text{or}$$

$$\boxed{\vec{x}_3 = \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$$

$\Rightarrow \{\vec{x}_2, \vec{x}_3\}$ forms a basis of E_{λ_2}

Example 3: Find the characteristic equation of ~~given~~ the matrix.

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$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: By definition, we know the characteristic equation is given by

$$\begin{aligned} 0 &= \det(A - \lambda I) = \det \left(\begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \\ &= \det \left(\begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix} \right) \\ &= (5-\lambda) \cdot (3-\lambda) (5-\lambda) (1-\lambda) \\ &= (5-\lambda)^2 (3-\lambda) (1-\lambda) \\ &= \lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 \end{aligned}$$

Note: It can be shown that for $A \in \mathbb{R}^{n \times n}$,

$$\det(A - \lambda I_n)$$

is an n th degree polynomial, called the characteristic polynomial of A

Note: For any polynomial of the form

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \dots + a_1\lambda + a_0$$

the companion matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}_{n \times n}$$

has characteristic polynomial $p(\lambda)$

□ Challenge: Can you prove this?

The characteristic equation $\det(A - \lambda I_n) = 0$ is an important theoretic tool for understanding eigenvalues.

- Translates a matrix equation $A\vec{x} = \lambda\vec{x}$ ← Unknown
Known Unknown

into a scalar equation $\det(A - \lambda I_n) = 0$

- Allows us to take advantage of the determinant function and its structure to compute eigenvalues
- Gives intuition for algorithms to compute eigenvalues

Downsides to the characteristic equation include

- Not practical from when trying to compute eigenvalues of general matrices (matrices with no special structure)
- Misleading since it seems to indicate we should solve for roots of a polynomial in order to find eigenvalues (NOT DONE IN PRACTICE)

If a root is unique (it's the only root with a specific numerical value)

we say that root has multiplicity one
 \Rightarrow simple root

If a root is non unique, ~~then we say~~
(more than one root has a particular numerical value), we say the root has multiplicity p ,
where p is the number of roots with the same value (called a multiple root)

Definition
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The algebraic multiplicity of an eigenvalue λ of $A \in \mathbb{R}^{n \times n}$ is ~~the~~ its multiplicity as a root ~~of~~ of the characteristic equation

Note:

□ Since we've argued

$$\det(A - \lambda I) = p(\lambda) \quad \text{characteristic polynomial}$$

$$= (-1)^n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0$$

we see

$$\deg(p(\lambda)) = n$$

□ By the fundamental theorem of Algebra, an n th order polynomial equation of the form $p(\lambda) = 0$ has precisely n -roots (real or complex) denoted as $\lambda_1, \lambda_2, \dots, \lambda_n$

□ We can write

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

These are the Eigenvalues of A

Section 5.3 : Diagonalization

Definition A square matrix A is said to be diagonalizable if $A \sim D$ for a diagonal matrix D .

Example: Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Lets find eigenvalues λ_1, λ_2

$$0 = \det(A - \lambda I) = \det \left(\begin{bmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{bmatrix} \right)$$

$$= (\lambda - 7)(\lambda - 1) + 8$$

$$= \lambda^2 - 8\lambda + 7 + 8$$

$$= \lambda^2 - 8\lambda + 15$$

$$= (\lambda - 5)(\lambda - 3)$$

$$\Rightarrow \lambda_1 = 5 \quad \text{and} \quad \lambda_2 = 3$$

Now lets find eigenvectors \vec{x}_1 and \vec{x}_2 .

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (A - \lambda_1 I_2) \vec{x}_1 = \begin{bmatrix} 7-5 & 2 \\ -4 & 1-5 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{x}_1}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (A - \lambda_2 I_2) \vec{x}_2 = \begin{bmatrix} 7-3 & 2 \\ -4 & 1-3 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \vec{x}_2}$$

$$\text{Let } X = [\vec{x}_1 \mid \vec{x}_2]$$

Hence we have

$$AX = A \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ -5 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= X D$$

$$\text{where } D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow AX = XD$$

$$\Rightarrow X^{-1}AX = D$$

$$\Rightarrow A \sim D \Rightarrow \boxed{A \text{ diagonalizable}}$$

Note: In previous example

$$X = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \Rightarrow X^{-1} = \frac{1}{-2+1} \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow X^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Thus we see

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = \underset{x}{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}} \underset{D}{\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}} \underset{x^{-1}}{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}.$$