

Lesson 26: Eigenstuff

Recall our differential equation from 2-mass, 3-spring system

$$M \ddot{u} = -K u$$

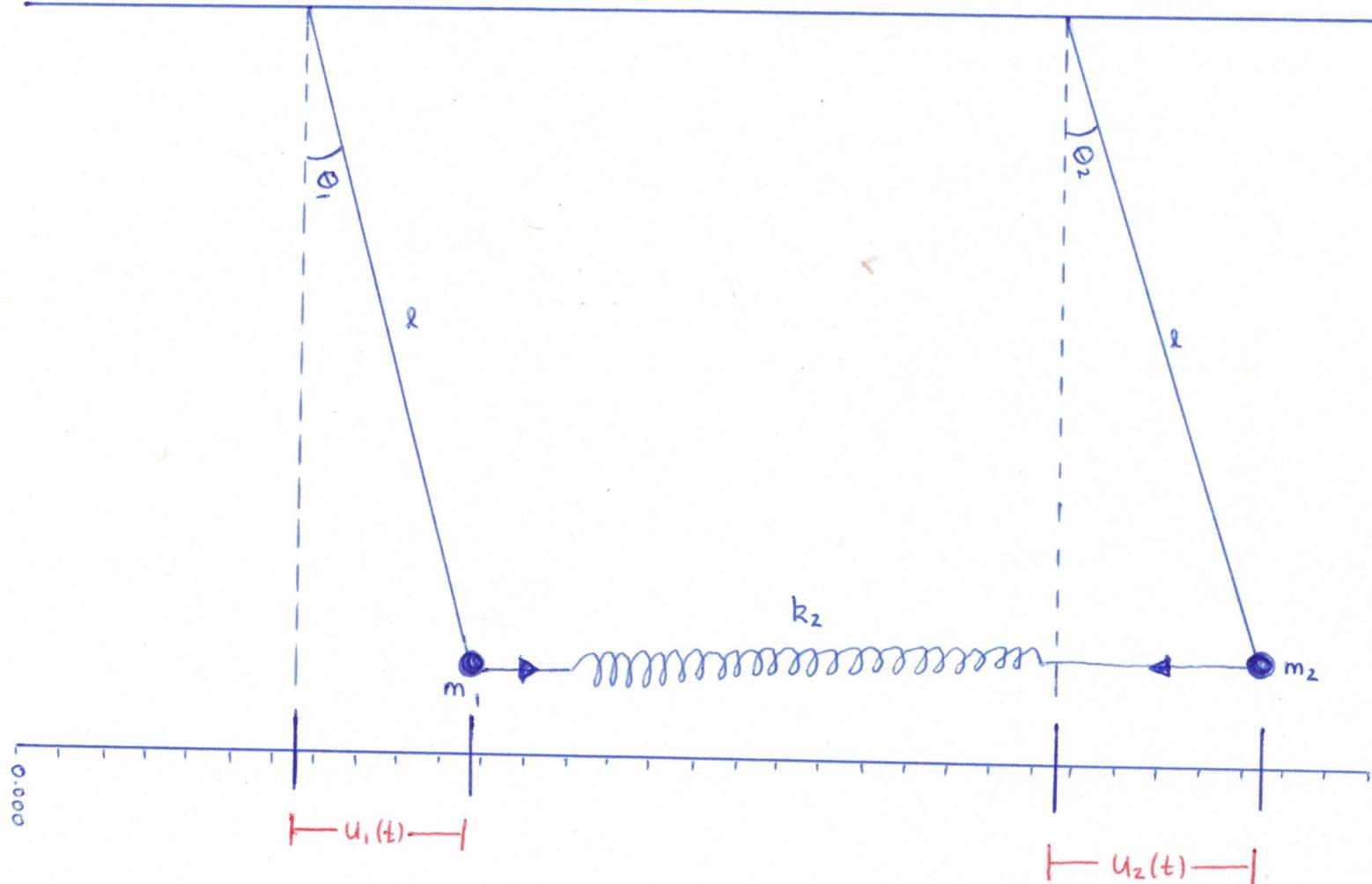
$$\Leftrightarrow \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} = - \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 + K_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Recall:

- We derived this differential Equation in Lesson 9: Matrix-Vector Equations
- You can find reminders of this derivation process in solutions to Sample Exam 1 VIA or V2A.

We call this the Undamped, unforced Oscillator Equation

Derivation of Differential Equations



spring constant for center spring

$$F_{S2} = k_2 \underbrace{(u_2 - u_1)}_{\text{elongation of spring}}$$

force internal
to spring

Assumption:

- spring has no mass

- Spring motion is modeled by $F_{S2} = k_2 \cdot e$
(zero intercept value - ideal spring)
- Motion does not loose energy.

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Today we will see an example of ~~this~~ a phenomenon
that can be modeled by this equation

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} = - \begin{bmatrix} \frac{m_1 g}{l} + k_2 & -k_2 \\ -k_2 & \frac{m_2 g}{l} + k_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Matrix Method for Solution: Use Cosine Ansatz

$$M \ddot{u} = -K u$$

$$\Rightarrow \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} = - \begin{bmatrix} \frac{m_1 g}{l} + k_2 & -k_2 \\ -k_2 & k_2 + \frac{m_2 g}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Make simplifying assumption that

$$m_1 = m_2 = m$$

$$\Rightarrow \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} = - \begin{bmatrix} \frac{mg}{l} + k_2 & -k_2 \\ -k_2 & k_2 + \frac{mg}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} = - \begin{bmatrix} \frac{g}{l} + \frac{k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2}{m} + \frac{g}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\ddot{u} = -M^{-1} K u$$

$$u(t) = A \cdot \cos(\omega t + \phi)$$

A = amplitude

ω = angular frequency

ϕ = phase

Note:

$$\omega = \frac{2\pi}{T} = 2\pi f$$

$f = \frac{1}{T}$ is ordinary frequency
measured in Hz

T = period of one cycle
measured in seconds

- Let's consider effect of the phase shift ϕ : the graph of $A \cos(\omega t + \phi)$

will cross the t -axis when

$$\cos(\omega t + \phi) = 0$$

$$\Rightarrow \omega t + \phi = \pi/2$$

$$\Rightarrow \omega t = \frac{\pi}{2} - \phi$$

$$\Rightarrow t = \frac{\frac{\pi}{2} - \phi}{\omega}$$

the graph of $\cos(\omega t)$ is shifted so that it crosses the t -axis at this value

Let $\vec{u}(t) = \underbrace{\vec{v} \cdot \cos(\omega t + \phi)}_{\text{cosine ansatz}} \quad \text{for unknown } \vec{v} \in \mathbb{R}^2$

comes from knowledge
of the physical system

$$= \cos(\omega t + \phi) \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad v_1, v_2 \in \mathbb{R}$$

$$= \begin{bmatrix} v_1 \cdot \cos(\omega t + \phi) \\ v_2 \cdot \cos(\omega t + \phi) \end{bmatrix}$$

$$\Rightarrow \ddot{u}(t) = \frac{d^2}{dt^2} \begin{bmatrix} \vec{u}(t) \end{bmatrix}$$

$$= \frac{d^2}{dt^2} \begin{bmatrix} v_1 \cdot \cos(\omega t + \phi) \\ v_2 \cdot \cos(\omega t + \phi) \end{bmatrix}$$

$$= -\omega^2 \cos(\omega t + \phi) \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Side Note:

$$\square \frac{d}{dt} [v_i \cdot \cos(\omega t + \phi)]$$

$$= v_i \cdot \frac{d}{dt} [\cos(\omega t + \phi)]$$

$$= v_i \cdot -\sin(\omega t + \phi) \cdot \omega$$

$$= -\omega v_i \sin(\omega t + \phi)$$

$$\square \frac{d^2}{dt^2} [v_i \cdot \cos(\omega t + \phi)]$$

$$= \frac{d}{dt} [-\omega v_i \sin(\omega t + \phi)]$$

$$= -\omega v_i \frac{d}{dt} [\sin(\omega t + \phi)]$$

$$= -\omega^2 \cdot v_i \cdot \cos(\omega t + \phi)$$

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$$\Rightarrow -\omega^2 \cos(\omega t + \phi) \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = - \begin{bmatrix} \frac{g}{l} + \frac{k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2}{m} + \frac{g}{l} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cdot \cos(\omega t + \phi)$$

$$\Rightarrow - \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cdot \cos(\omega t + \phi) = - \begin{bmatrix} \frac{g}{l} + \frac{k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2}{m} + \frac{g}{l} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cdot \cos(\omega t + \phi)$$

Bring to RHS and set LHS equal to zero.

$$\Rightarrow \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cdot \cos(\omega t + \phi) - \begin{bmatrix} \frac{g}{l} + \frac{k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2}{m} + \frac{g}{l} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cdot \cos(\omega t + \phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{bmatrix} - \begin{bmatrix} \frac{g}{l} + \frac{k_2}{m} & -\frac{k_2}{m} \\ -\frac{k_2}{m} & \frac{k_2}{m} + \frac{g}{l} \end{bmatrix} \right) \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cdot \cos(\omega t + \phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \omega^2 - \left(\frac{g}{l} + \frac{k_2}{m} \right) & \frac{k_2}{m} \\ \frac{k_2}{m} & \omega^2 - \left(\frac{k_2}{m} + \frac{g}{l} \right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \cdot \cos(\omega t + \phi) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \omega^2 - \left(\frac{g}{l} + \frac{k_2}{m}\right) & \frac{k_2}{m} \\ \frac{k_2}{m} & \omega^2 - \left(\frac{g}{l} + \frac{k_2}{m}\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \underbrace{\cos(\omega t + \phi)}_{} = 0$$

\tilde{A}

Not true for all values of t

$$\Rightarrow \tilde{A} \tilde{v} = \tilde{0} \quad \text{for non-zero } \tilde{v} \in \mathbb{R}^2$$

$$\Rightarrow \text{Nul}(\tilde{A}) \neq \{\tilde{0}\}$$

$$\Rightarrow \det(\tilde{A}) = 0 \quad \leftarrow \text{this is called the characteristic equation}$$

$$\Rightarrow \left[\omega^2 - \left(\frac{g}{l} + \frac{k_2}{m} \right) \right]^2 - \left[\frac{k_2}{m} \right]^2 = 0 \quad \leftarrow \text{this is known as the characteristic polynomial}$$

$$\Rightarrow \left[\omega^2 - \left(\frac{g}{l} + \frac{k_2}{m} \right) \right]^2 = \left[\frac{k_2}{m} \right]^2 \quad \text{take sq. root of both sides}$$

$$\Rightarrow \left| \omega^2 - \left(\frac{g}{l} + \frac{k_2}{m} \right) \right| = \frac{k_2}{m}$$

$$\Rightarrow \omega^2 - \left(\frac{g}{l} + \frac{k_2}{m} \right) = \pm \frac{k_2}{m}$$

Eigenvalue solutions: Eigenvalue 1 and associated Eigen vector

$$\omega_i^2 - \left(\frac{g}{l} + \frac{k_2}{m} \right) = -\frac{k_2}{m}$$

$$\Rightarrow \omega_1^2 = \frac{g}{l} + \frac{k_2}{m} - \frac{R_2}{m}$$

$$\Rightarrow w_i^2 = \frac{g}{\ell}$$

$$\Rightarrow \omega_1 = \sqrt{\frac{g}{\ell}} = \lambda_1$$

↑
Angular frequency
of cosine function
↑
eigenvalue of linear transformation

$$\Rightarrow \tilde{A} \tilde{v}_i = 0 \Leftrightarrow \begin{bmatrix} -\frac{k_2}{m} & \frac{k_2}{m} \\ \frac{k_2}{m} & -\frac{k_2}{m} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix}$$

eigenVector v_1 associated w/

$$\Rightarrow u_i(t) = A_i \cos(\omega_i t + \phi) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{eigenvalue } \lambda_1 = \omega_i$$

amplitude of oscillation phase delay 1 eigenvalue $\lambda_1 = \omega_i$

natural frequency of normal mode 1 (angular frequency ω_i) eigenvector 1

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Eigenvalue Solutions: Eigenvalue 2 and Associated Eigenvector

$$\omega_2^2 - \left(\frac{g}{\ell} + \frac{k_2}{m} \right) = + \frac{k_2}{m}$$

$$\Rightarrow \omega_2^2 = \left(\frac{g}{\ell} + \frac{k_2}{m} \right) + \frac{k_2}{m}$$

$$\Rightarrow \omega_2 = \sqrt{\frac{g}{\ell} + \frac{2k_2}{m}} = \lambda_2$$

Angular Frequency
of cosine function

eigenvalue of linear
transformation

$$\Rightarrow \tilde{A} \vec{v}_2 = \vec{0} \Leftrightarrow \begin{bmatrix} \frac{k_2}{m} & \frac{k_2}{m} \\ \frac{k_2}{m} & \frac{k_2}{m} \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Leftrightarrow \vec{U}_2(t) = A_2 \cos(\omega_2 t + \phi_2) \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

amplitude of oscillation
 phase delay 2
 frequency of resonant normal mode 2
 (angular frequency)

eigenvector 2

* For more about normal modes, see Wikipedia Article "Normal Mode"