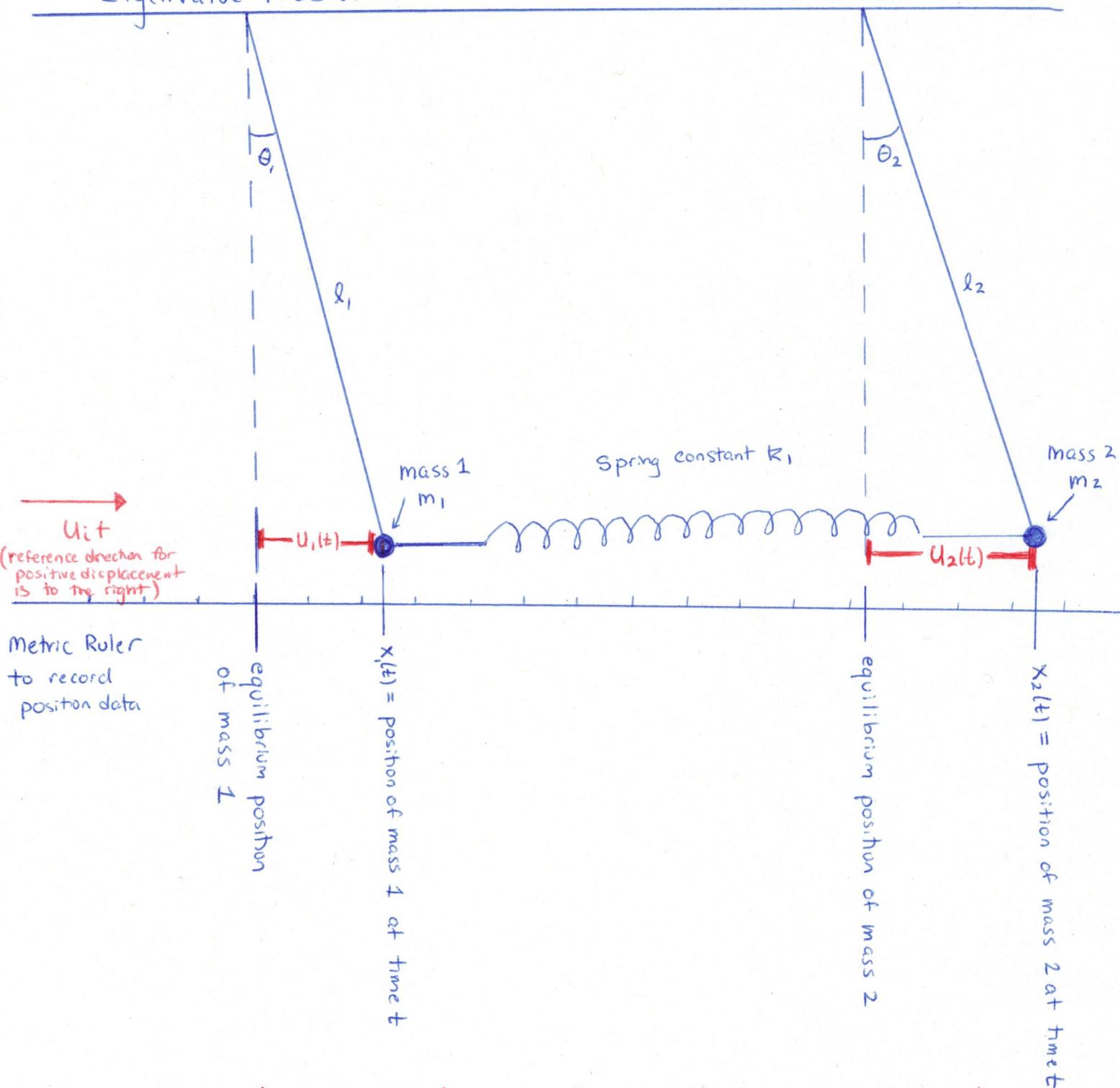


Derivation of 2nd Order ODE: $M\ddot{u} + Ku = 0$

- Modeling coupled pendula w/ connecting spring leading to Symmetric Eigenvalue Problem



$u_1(t)$ = displacement of mass 1 from equilibrium position at time t

$u_2(t)$ = displacement of mass 2 from equilibrium position at time t

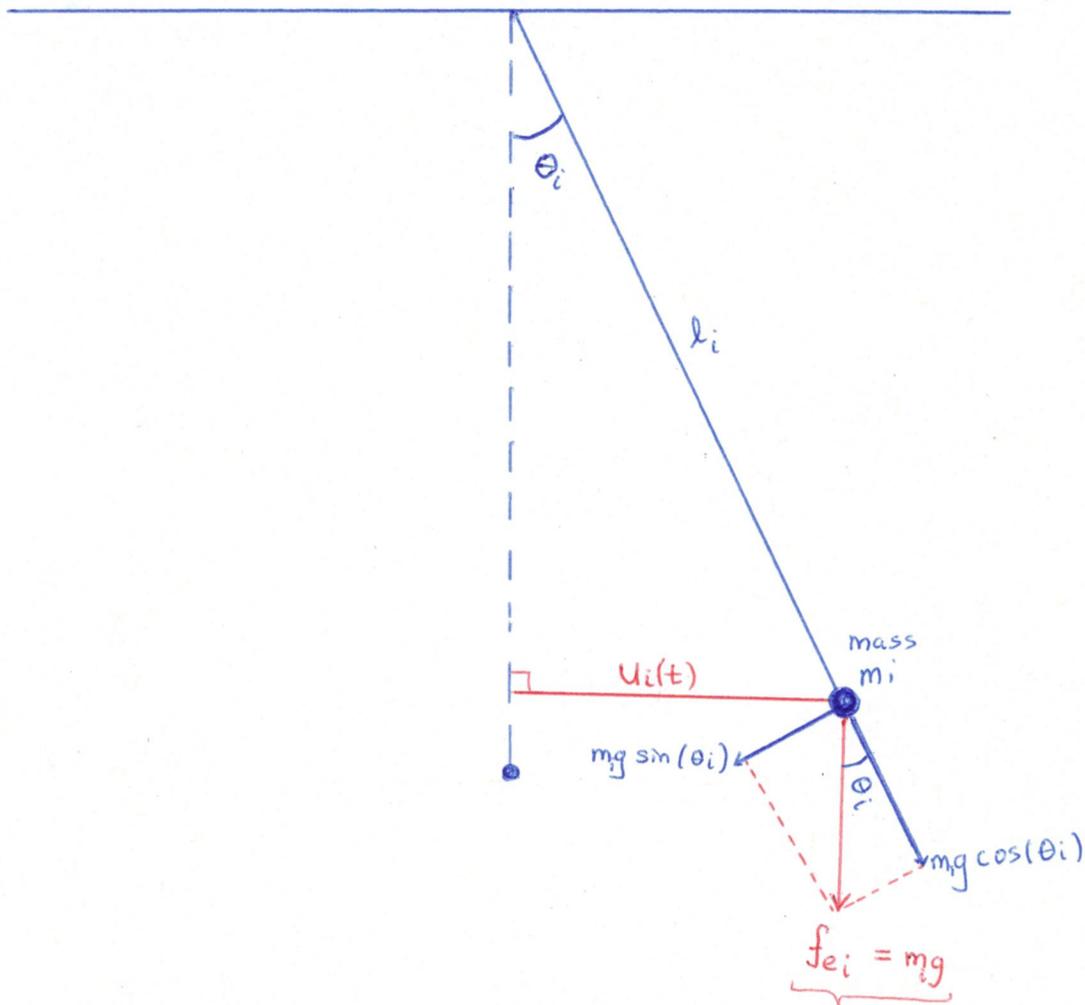
Let's recall our physical system from our in-class meeting and the ideal diagram from page 1 of these notes. For this system we will make the following simplifying assumptions

- the spring in this system has zero mass
- the internal force of the spring is modeled by an ideal version of Hooke's Law

$$F_{s_2} = k_2 e = k_2 (u_2 - u_1)$$

internal force of spring spring constant elongation of spring

- this system does not lose energy over time (aka no damping occurs over time)
- the length of the pendula are equal ($l_1 = l_2 = l$)



$f_{ei} = m_i g$
force of earth's gravity on point mass m_i at end of pendulum i

Newton's 2nd Law:

θ_i and associated acceleration $a_i(t)$ of mass m_i are in opposite direction

$\sum F_i$
sum of forces in tangential axis of point mass i

$$= - m_i g \sin(\theta_i) = m_i \cdot a_i(t) \quad \boxed{I}$$

magnitude of force on mass m_i

mass m_i of point mass on pendulum i

acceleration of point mass i in tangential axis

Now let's relate the acceleration $a_i(t)$ in the "tangential" axis to changes in angle θ_i using our arclength formula from Integral Calculus:

To this end, ~~consider~~ let

$a_i(t)$ = acceleration of mass m_i in axis tangent to inscribed arc of pendulum i motion

$\theta_i(t)$ = angle that rod i from pendulum i makes with vertical reference position.

Recall our arclength formula of an arc inscribed by a constant radius of length l_i is

$$s(t) = s = l_i \cdot \theta_i(t)$$

arc length radius angle

Then the rate of change of arclength is given by

$$\begin{aligned} v_i(t) &= \dot{s}(t) \\ &= \frac{d}{dt} [s(t)] \\ &= \frac{d}{dt} [l_i \cdot \theta_i(t)] \end{aligned} \quad \begin{aligned} &= l_i \cdot \frac{d}{dt} [\theta_i(t)] \\ &= \boxed{l_i \cdot \dot{\theta}_i(t)} \end{aligned}$$

We can find the acceleration of point mass in tangent axis as follows

$$a_i(t) = \ddot{s}(t)$$

$$= \frac{d^2}{dt^2} [s(t)]$$

$$= \frac{d}{dt} [\dot{s}(t)]$$

$$= \frac{d}{dt} [l_i \cdot \dot{\theta}_i(t)]$$

recall: $\dot{\theta}_i(t) = \frac{d}{dt} [\theta_i(t)]$

$$= l_i \frac{d}{dt} [\dot{\theta}_i(t)]$$

$$= \boxed{l_i \cdot \ddot{\theta}_i(t)} \quad \boxed{\text{II}}$$

$$= l_i \frac{d^2}{dt^2} [\theta_i(t)]$$

Combining formula $\boxed{\text{I}}$ & $\boxed{\text{II}}$ we see

$$m_i \cdot a_i(t) = -m_i g \sin(\theta_i)$$

$$\Rightarrow m_i \cdot \underbrace{l_i \ddot{\theta}_i(t)} = -m_i g \sin(\theta_i)$$

$$\Rightarrow \ddot{\theta}_i(t) = \frac{-g}{l} \sin(\theta_i)$$

Now we can state our differential equation as

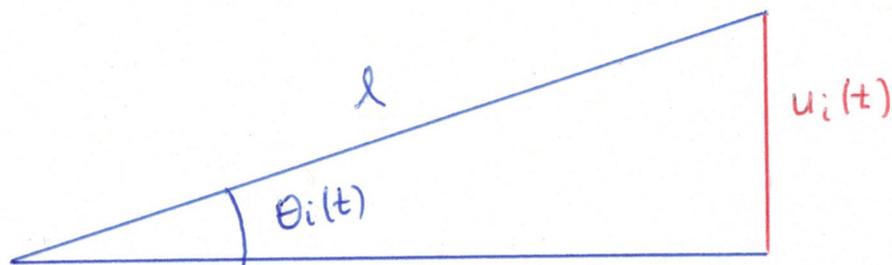
$$\ddot{\theta}_i(t) + \frac{g}{l} \sin(\theta_i) = 0 \quad \boxed{\text{III}}$$

Notice this differential equation is written w/ respect to angle θ_i but our desired displacement function $u_i(t)$ measures distances. We need to state both of these quantities using the same coordinate system. To do so, we want to transform our equation

$$\ddot{\theta}_i + \frac{g}{l} \sin(\theta_i) = 0$$

into rectangular coordinates so that the model of spring motion and the model of pendulum ~~motion~~ motion are ~~with~~ written with regard to the same frame of reference.

To this end, consider the right triangle



We know by our study of trigonometry that

$$\sin(\theta_i(t)) = \frac{u_i(t)}{l}$$

$$\Rightarrow \frac{u_i(t)}{l} = \sin(\theta_i(t))$$

Let's linearize the function $\sin(\theta_i(t))$. In particular, recall from our discussion of Taylor Series polynomials that

$$\sin(\theta_i(t)) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} [\theta_i]^{2k+1}$$

$$= \theta_i + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} \theta_i^{2k+1}$$

$$= \theta_i + \underbrace{\mathcal{O}(\theta_i^3)}_{\text{"big O" notation}}$$

Thus, we approximate $\sin(\theta_i) \approx \theta_i$ for "small" angles θ_i . (Note: an interesting follow up question is what does "small" mean here?)

Now, every where we see $\sin(\theta_i)$ in our equation III

we can use the approximation $\sin(\theta_i) \approx \theta_i$

$$\Rightarrow 0 = \ddot{\theta}_i(t) + \frac{g}{l} \sin(\theta_i) \approx \ddot{\theta}_i(t) + \frac{g}{l} \cdot \theta_i(t) = 0$$

$$\Rightarrow \ddot{\theta}_i(t) + \frac{g}{l} \theta_i(t) = 0 \quad \text{is a linear approximation to our nonlinear ODE III}$$

Using the sine relationship that

$$\frac{u_i(t)}{l} = \sin(\theta_i(t)) \approx \theta_i(t)$$

We see we can ~~write the~~ approximate the motion of our pendulum as

$$\frac{\ddot{u}_i(t)}{l} + \frac{g}{l} \left[\frac{u_i(t)}{l} \right] = 0$$

$$\Rightarrow \frac{\ddot{u}_i(t)}{l} + \frac{g}{l^2} u_i(t) = 0$$

$$\Rightarrow \ddot{u}_i + \frac{g}{l} u_i = 0$$

$$\Rightarrow \ddot{u}_i = -\frac{g}{l} u_i$$

$$\Rightarrow m_i \cdot \ddot{u}_i = -\frac{m_i \cdot g \cdot u_i}{l}$$

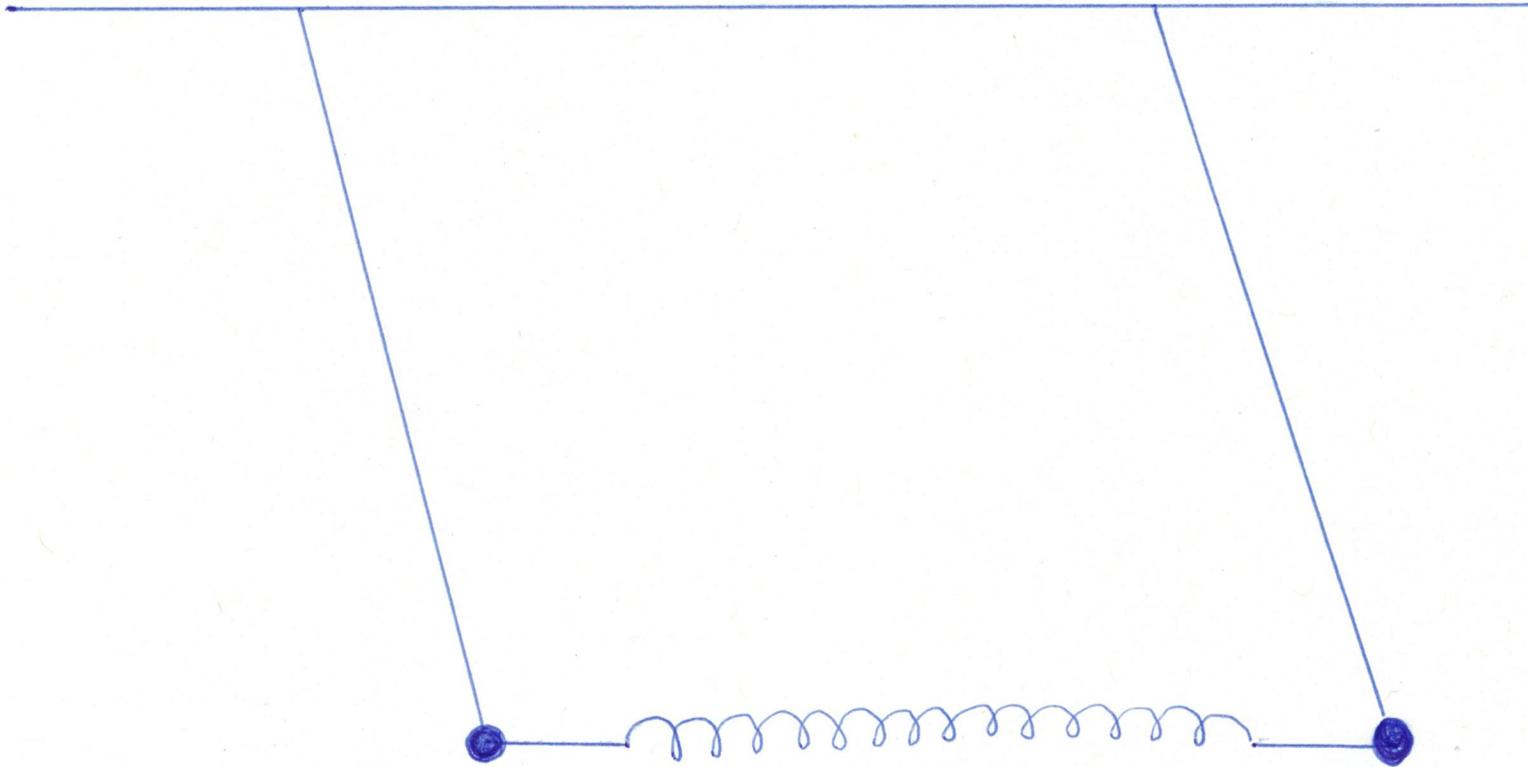
linear approximation

of force exerted on
mass 1 as part of
dynamic of pendulum

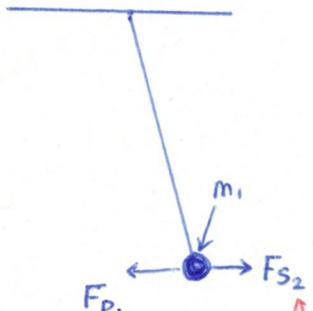
$$\Rightarrow \boxed{F_{Pi} = -\frac{m_i g}{l} \cdot u_i}$$

Now, let's use this approximation combined with our knowledge of Hooke's Law to derive our 2nd order differential equation for our original system.

Free Body Diagrams for Mass 1 & Mass 2



Mass 1 : $\sum F_{m_1} = F_{p_1} + F_{s_2}$

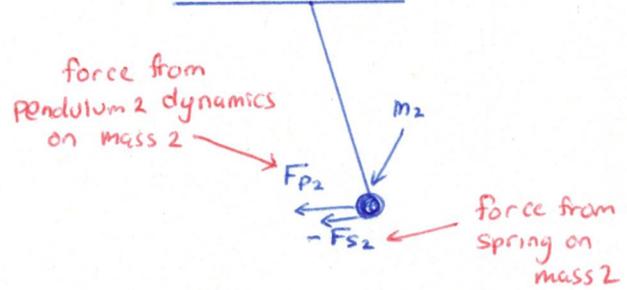


force from pendulum 1 dynamics exerted on mass 1 (includes a negative sign)

force from spring exerted on mass 1

see page ③ for details

Mass 2:



$\sum F_{m_2} = F_{p_2} - F_{s_2}$

$$\begin{bmatrix} \sum F_{m_1} \\ \sum F_{m_2} \end{bmatrix} = \begin{bmatrix} F_{p_1} + F_{s_2} \\ F_{p_2} - F_{s_2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{m_1 g}{l} \cdot u_1 + k_2 (u_2 - u_1) \\ -\frac{m_2 g}{l} u_2 + -k_2 (u_2 - u_1) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{m_1 g}{l} u_1 - k_2 u_1 + k_2 u_2 \\ k_2 u_1 - k_2 u_2 - \frac{m_2 g}{l} u_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{m_1 g}{l} - k_2 & k_2 \\ k_2 & -k_2 - \frac{m_2 g}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

2×2 2×1

$$= \underbrace{\begin{bmatrix} \frac{m_1 g}{l} + k_2 & -k_2 \\ -k_2 & k_2 + \frac{m_2 g}{l} \end{bmatrix}}_K \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\vec{u}}$$

By Newton's 2nd Law, we know

$$\begin{bmatrix} \sum F_{m_1} \\ \sum F_{m_2} \end{bmatrix} = \begin{bmatrix} m_1 \ddot{u}_1 \\ m_2 \ddot{u}_2 \end{bmatrix}$$

$$= \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix}$$

$$= M \ddot{u}$$

$$\Rightarrow M \ddot{u} = -K u$$

$$\Rightarrow M \ddot{u} + K u = \vec{0} \quad \text{with}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad K = \begin{bmatrix} \frac{m_1 g}{l} + k_2 & -k_2 \\ -k_2 & k_2 + \frac{m_2 g}{l} \end{bmatrix}$$

$$\text{and } \ddot{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$