Definition 9.1: Vector Space

A **vector space** is a nonempty set V of objects, called vectors, equipped with two operations:

i. *Vector addition*:

   Adding any pair of vectors $\mathbf{v}, \mathbf{w} \in V$ gives another vector $\mathbf{v} + \mathbf{w} \in V$
   (closed under vector addition)

ii. *Scalar multiplication*:

   Multiplying any vector $\mathbf{v} \in V$ by a scalar $c \in \mathbb{R}$ gives another vector $c\mathbf{v} \in V$
   (closed under scalar multiplication)

For a space to be a vector space, the two operations listed above must satisfy a number of properties. In particular, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalars $c, d \in \mathbb{R}$, we must have:

1. *Commutativity of vector addition*: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
2. *Associativity of vector addition*: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
3. *Additive identity*: There is a zero element $\mathbf{0} \in V$ such that
$$\mathbf{u} + \mathbf{0} = \mathbf{u} = \mathbf{0} + \mathbf{u}.$$
4. *Additive inverse*: For each $\mathbf{u} \in V$ there is a $-\mathbf{u} \in V$ such that
$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0} = (-\mathbf{u}) + \mathbf{u}.$$
5. *Distributivity over vector addition*: $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
6. *Distributivity over scalar addition*: $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
7. *Associativity of scalar multiplication*: $c(d\mathbf{u}) = (cd)\mathbf{u}$.
8. *Multiplicative identity of scalar multiplication*: the scalar $1 \in \mathbb{R}$ satisfies $1\mathbf{u} = \mathbf{u}$.

EXAMPLE 9.1.1

The quintessential example of a real vector space is the set $\mathbb{R}^m$ equipped with vector addition and scalar multiplication we defined previously. In particular, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we define

$$\mathbf{x}+\mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_m + y_m \end{bmatrix}, \qquad\qquad c\mathbf{x} \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_m \end{bmatrix}.$$

The zero vector $\mathbf{0} \in \mathbb{R}^m$ is the $m \times 1$ column vector with all zero entries. Moreover, we confirmed that $\mathbb{R}^n$ satisfies all of the algebraic properties in Theorem 9 during our discussion of vector operations. Notice, we do not consider the multiplication between vectors (inner products) when studying the vector space $\mathbb{R}^n$. This is a general theme of all vector spaces: we focus only on vector-vector addition and scalar-vector multiplication in our study of vector spaces. Any other operations between vectors may be helpful for solving problems but no other operations is part to the definition of general vector spaces.

EXAMPLE 9.1.2

Recall $R^{m\times n}$ denotes the set of all $m \times n$ matrices with real-valued entries. Given the operations of matrix addition and scalar-matrix multiplication, we confirm that $\mathbb{R}^{m\times n}$ forms a real vector space. The zero "vector" of this space is the zero matrix

$$0 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

Notice that $\mathbb{R}^m$ $\mathbb{R}^{m\times 1}$ is a special case of a vector space of matrices with 1 column. Once again, we do not consider matrix-matrix multiplication when studying the vector space $\mathbb{R}^{m\times n}$. Instead, we focus only on matrix addition and scalar-matrix multiplication.

As we have seen, our solutions to the matrix-vector multiplication problem and the linear-systems problem depend heavily on arithmetic. These problems, and their solutions, very much depend on the algebra of vectors spaces $\mathbb{R}^{m\times n}$ and $\mathbb{R}^n$. We very much value this reliance because large-scale computers can very accurately execute addition and multiplication in these spaces. Thus, as long as we study the properties of these vectors spaces and maps between these vectors spaces, we can generate programmable algorithms for computer execution. This then enables us to solve any problem from these classes using computers. This is a very powerful tool to have in our tool bag.

However, as we will see, not all problems can be stated as either matrix-vector multiplication problems or linear-systems problems. In particular, there is a huge class of problems that require more sophisticated technology to state and solve. In fact, many problems from the fields of differential equations, numerical methods, and Fourier analysis depend on vectors spaces of functions. The difference between $\mathbb{R}^{m\times n}$ and a vector space of functions V is analogous to the difference between arithmetic and calculus. In arithmetic, we are studying the algebraic properties of addition and multiplication of numbers. By contrast, in calculus, we are studying properties of functions and continuity. We pass from the discrete into the continuous, increasing capacity at the cost of simplicity.

From this standpoint, we begin our departure from the world of matrices and begin our ascent to the world of functions. Our goal is to develop a number of useful vector spaces of functions that we can then use to solve differential equations, approximate noisy data and interpolate sampled data points. We will use this functionality to solve least-squares problems and eigenvalue problems which arise in mathematical modeling processes.

EXAMPLE 9.1.3

Recall from our discussion of Important Sets of Functions, we defined

$$P_n(I) = \{f \, : \, f : I \to \mathbb{R} \text{ and } f \text{ is a polynomial with deg } (f) \leq n \text{ for } n \in \mathbb{N}\}$$

where $I \subseteq \mathbb{R}$. As we will see, this space forms a vector space under the operation of polynomial addition and scalar-polynomial multiplication. Let $f, g \in P_n(I)$. By definition, we know

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \\ g(x) &= b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0. \end{aligned}$$

where $a_i, b_i \in \mathbb{R}$ for all $i = 1, 2, ..., n$. Then, we define the polynomial addition for this vector space as follows

$$f + g = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0)$$

We confirm that the sum of polynomials $f(x)$ and $g(x)$ has degree $\leq n$ and thus $P_n(I)$ is closed under polynomial addition. We can also define scalar-polynomial multiplication. For $f \in P_n(I)$ and $c \in \mathbb{R}$, we set

$$c \cdot f = (ca_n)x^n + (ca_{n-1})x^{n-1} + \cdots + (ca_1)x + ca_0$$

which is a polynomial of degree $\leq n$.

Using the definition of the set $P_n(I)$ and our operations of polynomial addition and scalar-polynomial multiplication, we can verify the 8 algebraic axioms of vectors spaces. By doing so, we confirm that $P_n(I)$ is indeed a vector space.

Remarks:

- The set of polynomials of degree exactly equal to n is not a vector space. Indeed, consider the following counter example:

$$\underbrace{(x^2 + 3x + 4)}_{f(x)} + \underbrace{(2 - x^2)}_{g(x)} = 3x + 6$$

  Here, we add two polynomials of degree equal to 2 and produce a polynomial of degree 1. Thus, the set of polynomials of degree equal to 2 is not closed under polynomial addition. By extension, this same statement is true for the set of polynomials equal to n.

- Be very careful with notation. The difference between scalars and constant polynomials is substantial and important. However, the notation we use is identical. For example the zero polynomial is denotes as $f = 0$ while zero as a real number is given as 0. The first of these represents the constant function whose output value is 0 for all $x \in I$. The second denotes one number 0.

- When we talk about $P_n(I)$ as a vector space, we refer to each of the elements of this set as vectors. In other words, from the context of vector spaces, polynomials are considered vectors. This is a major paradigm shift in this textbook. Prior to this chapter, our entire discussion of vectors has a very specific meaning: vectors were organized lists of real numbers. However, from this point forward, we generalize our definition of vectors. In this context, vectors are simply elements of vector spaces. Thus, for $P_n(I)$, vectors are functions.

Theorem 37: Algebraic Identities for Vectors Spaces

The following identities are consequences of the properties above:

I. $0\mathbf{u} = \mathbf{0}$:

II. $-\mathbf{u} = -1\mathbf{u}$:

III. $c\,\mathbf{0} = \mathbf{0}$:

IV. If $c\,\mathbf{v} = \mathbf{v}$, then either $c = 0$ or $\mathbf{v} = 0$.

Proof. Let V be a vector space over $\mathbb{R}$. Let's begin with the first proposition. To this end, let $\mathbf{u} \in V$. We want to show $0 \cdot \mathbf{u} = 0$. To this end, let $\mathbf{v} = 0 \cdot \mathbf{u}$ and consider

$$\begin{aligned} \mathbf{v} + \mathbf{v} &= 0 \cdot \mathbf{u} + 0 \cdot \mathbf{u} \\ &= (0 + 0) \cdot \mathbf{u} \\ &= 0 \cdot \mathbf{u} \\ &= \mathbf{v} \end{aligned}$$

However, since $\mathbf{v} \in V$, we know $-\mathbf{v} \in V$ and we can consider

$$\mathbf{v} + \mathbf{v} + -\mathbf{v} = \mathbf{v} + -\mathbf{v} \qquad \Longrightarrow \qquad \mathbf{v} = \mathbf{0}$$

This is what we wanted to show.

Next, let's show that $-1 \cdot \mathbf{u} = -\mathbf{u}$. To this end, let $-1 \in \mathbb{R}$ and $\mathbf{u} \in V$. Consider

$$(1 + -1) \cdot \mathbf{u} = 0 \cdot \mathbf{u} = \mathbf{0}$$

However, we know by distributivity over scalar addition that

$$\begin{aligned} (1 + -1) \cdot \mathbf{u} &= 1 \cdot \mathbf{u} + -1 \cdot \mathbf{u} \\ &= \mathbf{u} + -1 \cdot \mathbf{u} \end{aligned}$$

where the second line results from the multiplicative identity property of scalar multiplication. Thus, combining these together we have

$$\mathbf{u} + -1 \cdot \mathbf{u} = \mathbf{0}$$

Since $\mathbf{u} \in V$, we know there exists an additive inverse $-\mathbf{u} \in V$. Then, consider

$$-\mathbf{u} + \mathbf{u} + -1 \cdot \mathbf{u} = -\mathbf{u} + \mathbf{0} \qquad \Longrightarrow \qquad -1 \cdot \mathbf{u} = -\mathbf{u}$$

This is what we wanted to establish.

Finally, let's establish $c \cdot \mathbf{0} = \mathbf{0}$ for any $c \in \mathbb{R}$. To this end, let $c \in \mathbb{R}$ and let $\mathbf{0} \in V$ be the zero vector. Consider

$$\begin{aligned} c \cdot \mathbf{0} &= c \cdot (\mathbf{0} + \mathbf{0}) \\ &= c \cdot \mathbf{0} + c \cdot \mathbf{0} \end{aligned}$$

Since $c \cdot \mathbf{0} \in V$, so too is it's additive inverse $-c \cdot \mathbf{0}$. We can consider

$$\begin{aligned} && c \cdot \mathbf{0} + -c \cdot \mathbf{0} &= c \cdot \mathbf{0} + c \cdot \mathbf{0} + -c \cdot \mathbf{0} \\ \implies && (c + -c) \cdot \mathbf{0} &= c \cdot \mathbf{0} \\ \implies && 0 \cdot \mathbf{0} &= c \cdot \mathbf{0} \\ \implies && \mathbf{0} &= c \cdot \mathbf{0} \end{aligned}$$

This is what we needed to show. $\square$

The fourth part of the proof for the theorem above is left to the reader as an exercise.

Definition 9.2: Vector Space

A **subspace of a vector space** V is a nonempty set $W \subseteq V$ that has three properties:

i. *Zero vector*: $\mathbf{0} \in W$.

ii. *Closed under vector addition*: For each $\mathbf{u}, \mathbf{v} \in W$, the sum $\mathbf{u} + \mathbf{v} \in H$.

iii. *Closed under scalar multiplication*: For each $\mathbf{u} \in W$ and each scalar $c \in \mathbb{R}$, the vector $c\mathbf{u} \in W$.

Theorem 38: Check for Subspace: Option 1

A non-empty subset $W \subseteq V$ of a vector space V is a subspace if and only if

a. for every $\mathbf{u}, \mathbf{v} \in W$, the sum $\mathbf{u} + \mathbf{v} \in W$ and

b. for every $\mathbf{u} \in W$ and every $c \in \mathbb{R}$, the scalar product $c\mathbf{u} \in W$.

Theorem 39: Check for Subspace: Option 2

A non-empty subset $W \subseteq V$ of a vector space V is a subspace if and only if

a. for every $\mathbf{u}, \mathbf{v} \in W$ and $c, d \in \mathbb{R}$, the sum $c\,\mathbf{u} + d\,\mathbf{v} \in W$.

Theorem 40: Spans form Subspaces

If $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p$ are vectors in a vector space V, then the span of these vectors

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{c}_p \, : \, c_i \in \mathbb{R} \text{ for all } i \in \{1, 2, ..., p\}\}$$

Lesson 18: Vector Spaces- Suggested Problems

1. Exercise 4.1.1 p. 195
2. Exercise 4.1.2 p. 195-196
3. Exercise 4.1.5 p. 196
4. Exercise 4.1.6 p. 196
5. Exercise 4.1.9 p. 196
6. Exercise 4.1.13 p. 196
7. Exercise 4.1.20 p. 196
8. Exercise 4.1.21 p. 196

Lesson 18: Vector Spaces- Challenge Problems

1. Exercise 4.1.19 p. 196
2. Exercise 4.1.33 p. 197
3. Exercise 4.1.37 p. 198
4. Exercise 4.1.38 p. 198

