Section 8.7 : Taylor & Maclaurin Series

- ☐ Which functions have power series representations?
- ☐ How can we find these respresentations, if they exist?

Assume any function $f(x)$ can be represented as a power series

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

OR

$$f(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots$$

if $|x-a| < R$

Our task will be to try to determine coefficients c_n.

Major observation: If $f(x) = g(x)$, then
- $\frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}\left[g(x)\right]$
- $\frac{d^n}{dx^n}\left[f(x)\right] = \frac{d^n}{dx^n}\left[g(x)\right]$

At $x = a$: Evaluating $f(a)$

$$f(x) = f(a) = \sum_{n=0}^{\infty} c_n (x-a)^n \Bigg|_{x=0} = c_0 \Rightarrow c_0 = f(a)$$

Evaluating $f'(a)$:

$$f'(a) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right]\Bigg|_{x=a}$$

$$\cancel{\sum \frac{d}{dx}}$$

$$= \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n (x-a)^n\right]\Bigg|_{x=a}$$

$$= \sum_{n=1}^{\infty} n \cdot c_n (x-a)^{n-1} \Bigg|_{x=a}$$

Theorem 8.7.5 p. 605

If f has a power series representation (expansion) at a, ie if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{for} \quad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{d^n}{dx^n}\left[f(x)\right]\Big|_{x=a}$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \quad \text{where}$$

we call this a Taylor series expansion of the function

$f(x)$ at ~~a~~ centered at a.

If $a=0$, then $f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(0)}{n!} x^n$ is called a Maclauren Series.

Example 8.7.1 p. 606

Find the MacClaorin Series of $f(x) = e^x$ and its radius of convergence

Solution: If $f(x) = e^x$ $\Rightarrow \frac{d}{dx}[e^x] = e^x$

$$\Rightarrow \frac{d^n}{dx^n}[e^x] = e^x$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

To find the radius of convergence, let $a_n = \frac{x^n}{n!}$ and consider

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right|$$

$$= \left|\frac{x}{n+1}\right|$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 \quad \text{for all } x \in \mathbb{R} \Rightarrow R = \infty.$$

Example 3: Find Taylor series for $f(x) = e^x$ at $a = 2$.

Solution:

Example 8.7.7 p.611: for $f(x) = \sin(x)$, find Taylor series at $a = \pi/3$.

Example 8.7.4 p. 609

Find the Maclauren Series for $f(x) = \sin(x)$.

Cyclical Derivatives

Let $f(x) = \sin(x)$.

Consider

$$f(x) = \sin(x) \qquad f(0) = 0$$

$$f'(x) = \cos(x) \qquad f'(0) = 1$$

$$f''(x) = -\sin(x) \qquad f''(0) = 0$$

$$f'''(x) = -\cos(x) \qquad f'''(0) = -1$$

$$f^{(4)}(x) = \sin(x) \qquad f^{(4)}(0) = 0$$

$$\Rightarrow f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

$$= x + \frac{-x^3}{3!} + \frac{x^5}{5!} + \frac{-x^7}{7!} + \frac{x^9}{9!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Example 8.7.5 p. 610

Find the ~~Macct~~ Maclaurin Series for ~~$*$~~ $\cos(x)$.

Method 1: Use Definition

Let $f(x) = \cos(x)$

$n=0 \qquad f(x) = \cos(x) \Rightarrow f(0) = 1$

$n=1 \qquad f'(x) = -\sin(x) \Rightarrow f'(0) = 0$

$n=2 \qquad f''(x) = -\cos(x) \Rightarrow f''(0) = -$

$n=3 \qquad f'''(x) = \sin(x) \Rightarrow f'''(0) = 0$

$n=4 \qquad f^{(4)}(x) = \cos(x) \Rightarrow f^{(4)}(0) =$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-a)^n$$

$$= 1 + \frac{0}{1!}(x-0)^1 + \frac{-1}{2!}(x-0)^2 + \frac{0}{3!}(x-0)^3 + \frac{1}{4!}(x-$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Example 8.7.5 p. 610 continued ...

Method 2: Using Thm 8.6.2

$$\cos(x) = \frac{d}{dx}\left[\sin(x)\right]$$

$$= \frac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right] \qquad x \in \mathbb{R}$$

$$= \sum_{n=0}^{\infty} \frac{d}{dx}\left[(-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}\right]$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{d}{dx}\left[\frac{x^{2n+1}}{(2n+1)!}\right]$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n+1)!} \frac{d}{dx}\left[x^{2n+1}\right]$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n+1)!} \cdot (2n+1)\, x^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

□ If $f(x) \in C^{(\infty)}(D)$ for some $D \subseteq \mathbb{R}$, then when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad ?$$

$$= \lim_{N \to \infty} \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \lim_{N \to \infty} T_N(x)$$

where

$$T_N(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is a polynomial of degree N called the Nth degree Taylor polynomial of f at a.

Let's define $R_N(x)$ to be the remainder of the Taylor series $T_N(x)$ given by

$$R_N(x) = f(x) - T_N(x) \qquad \Rightarrow \quad f(x) = T_N(x) + R_N(x)$$

If we can show $\lim_{N\to\infty} R_N(x) = 0$ then we know

$$\lim_{N\to\infty} T_N(x) = \lim_{N\to\infty} f(x) - R_N(x)$$

$$= \lim_{N\to\infty} f(x) - \lim_{N\to\infty} R_N(x)$$

$$= f(x) - 0$$

$$= f(x) \ \checkmark$$

Theorem: If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor Series on the interval $|x-a| < R$.

Taylor's Inequality

If $|f^{(N+1)}(x)| \leq M$ for $|x-a| \leq d$,

then the remainder $R_N(x)$ of the Taylor Series

satisfies the inequality

$$|R_N(x)| \leq \frac{M}{(N+1)!} |x-a|^{N+1}$$

This Theorem Gaurantees Uniform convergence

for all $|x-a| \leq d$.

Outline of Proof: Consider $N=1$ and assume $|f^{(2)}(x)| = |f''(x)| \leq M$.

$$\Rightarrow \quad f''(x) \leq M \quad \text{for} \quad a \leq x \leq a+d$$

$$\Rightarrow \quad \int_a^x f''(t)\,dt \leq \int_a^x M\,dt$$

$$\Rightarrow \quad f'(x) - f'(a) \leq M(x-a)$$

$$\Rightarrow \quad f'(x) \leq f'(a) + M(x-a)$$

$$\Rightarrow \quad \int_a^x f'(t)\,dt \leq \int_a^x \left[f'(a) + M(t-a)\right] dt$$

$$\Rightarrow \quad f(x) - f(a) \le f'(a)(x-a) + M\frac{(x-a)^2}{2!}$$

$$\Rightarrow \quad f(x) - \underbrace{\left[f(a) + f'(a)(x-a)\right]} \le M\frac{(x-a)^2}{2!}$$

$$\Rightarrow \quad \underbrace{f(x) - T_1(x)} \le \frac{M(x-a)^2}{2!}$$

$$\Rightarrow \quad R_1(x) \le \frac{M(x-a)^2}{2!}$$

A similar argument shows $f''(x) \ge -M \Rightarrow R_1(x) \ge \frac{-M}{2}(x-a)^2$

$$\Rightarrow \quad |R_1(x)| \le \frac{M}{2}|x-a|^2$$

- We've shown this only for $x > a$ but we can use the same argument to show this holds for $x < a$.

- This shows case for $N = 1$. To show for N, we integrate $N+1$ times

Recall : $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges by ratio test

$$\Rightarrow \lim_{n\to\infty} \frac{x^n}{n!} = 0$$

$$\Rightarrow \lim_{N\to\infty} \left| R_N(x) \right| \le \lim_{N\to\infty} \frac{M}{(N+1)!} \left| x-a \right|^{N+1} = 0 \checkmark$$

$$\Rightarrow \lim_{N\to\infty} \left| R_N(x) \right| = 0$$ by squeeze theorem.

Example 8.7.2 p. 608

Prove that e^x is equal to the sum of it's Maclaurin Series

Solution: Recall that if $e^x = f(x) \Rightarrow f^{(N+1)}(x) = e^x$ for all N.

To show that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

we need to show

$$\lim_{N\to\infty} R_N(x) = 0.$$

To do this, we use our powerful

Taylor's Inequality

and a bit of Ingenuity!

We define $R_N(x) = e^x - T_N(x)$

$$= e^x - \sum_{n=0}^{N} \frac{x^n}{n!}$$

By Taylor's Inequality, if we can find a real number M such that

$$\left| f^{(N+1)}(x) \right| \leq M \quad \text{for all } |x-0| \leq d$$

then we can bound our error above.

For and $d \in \mathbb{R}^+$ and $|x-a| \leq d$, we know

$$|f^{(N+1)}(x)| = |e^x| = e^x \leq e^d$$

$$\Rightarrow \quad |R_N(x)| \leq \frac{e^d}{(N+1)!} |x|^{N+1} \quad \text{for} \quad |x| \leq d$$

$$\Rightarrow 0 \leq \lim_{N\to\infty} |R_N(x)| \leq \lim_{N\to\infty} \frac{e^d}{(N+1)!} |x|^{N+1}$$

$$= \cancel{ed}\; e^d \lim_{N\to\infty} \frac{|x|^{N+1}}{(N+1)!} = 0$$

$\Rightarrow \lim_{N\to\infty} |R_N(x)| = 0$ by squeeze thm

$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

$$\Rightarrow e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

To prove $\sin(x) \text{ "=" } \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x

we need to show $|R_N(x)| \to 0$ as $N \to \infty$.

Using Taylor's Inequality, we want to find M s.t.

$$|f^{(N+1)}(x)| \leq M = 1 \longrightarrow f^{(N+1)}(x) = \pm \sin(x) \text{ OR } f^{(N+1)}(x) = \pm \cos(x)$$

$$\Rightarrow |R_N(x)| \leq \frac{M}{(N+1)!} |x|^{N+1} = \frac{|x|^{N+1}}{(N+1)!} \longrightarrow 0 \quad \checkmark$$

Example 8.7.8 p. 611 : Find Maclaurin Series for $f(x) = (1+x)^K$

$f(x) = (1+x)^K$ $\qquad$ $f(0) = 1$

$f'(x) = K \cdot (1+x)^{K-1}$ $\qquad$ $f'(0) = K$

$f''(x) = K \cdot (K-1) \cdot (1+x)^{K-2}$ $\qquad$ $f''(0) = K \cdot (K-2)$

$f'''(x) = K \cdot (K-1) \cdot (K-2) \, (1+x)^{K-3}$ $\qquad$ $f''(0) = K \cdot (K-1)(K-2)$

$\vdots$

$f^{(n)}(x) = K \cdot (K-1) \cdots (K-n+1) \, (1+x)^{K-n}$ $\qquad$ $f^{(n)}(0) = K(K-1) \cdots (K-n+1)$

$$\Rightarrow \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{K \cdot (K-1) \cdot (K-2) \cdots (K-n+1)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \binom{K}{n} x^n$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k\cdot(k-1)\cdots(k-n)\,x^{n+1}}{(n+1)!}\right| \left|\frac{n!}{k(k-1)\cdot\cdots(k-n+1)\;x^n}\right|$$

$$= \frac{|k-n|}{n+1}\cdot|x|$$

$$= \frac{|n-k|}{n+1}\,|x|$$

$$= \frac{|1-\frac{k}{n}|}{|1+\frac{1}{n}|}\,|x| \to |x|$$

$$\Rightarrow \boxed{f(x) = \sum_{n=0}^{\infty}\binom{k}{n}x^n} \iff |x| < 1.$$

Binomial Series

Example 8.7.9 : For $f(x) = \dfrac{1}{\sqrt{4-x}}$, find the Maclaurin series & Radius of convergence.

$$\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4\left(1-\frac{x}{4}\right)}}$$

$$= \frac{1}{\sqrt{4}\cdot\left(1-\frac{x}{4}\right)^{1/2}}$$

$$= \frac{1}{2}\cdot\left(1-\frac{x}{4}\right)^{-1/2} \qquad K = -1/2, \quad x \to \frac{-x}{4}$$

$$= \frac{1}{2}\cdot\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(-\frac{x}{4}\right)^n$$

$$\boxed{= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\cdot\binom{-1/2}{n}\cdot\frac{1}{4^n}\cdot x^n}$$

expand for more!

$$= \frac{1}{2}\left[1 + \frac{-1}{2}\cdot\frac{x}{4}\right.$$

Example 8.7.10 p. 613: Find the sum

$$\frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^2} - \frac{1}{4\cdot 2^4} + \cdots$$

Solution:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n\cdot 2^n} = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}\cdot\left(\frac{1}{2}\right)^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{\left(\frac{1}{2}\right)^n}{n}$$

$$= \ln\left(1+\frac{1}{2}\right)$$

$$= \ln\left(3/2\right)$$