Definition 8.5.1
p. 592

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n$$

notice the bottom limit is 0 (NOT 1)

where x is a variable and the c_n's are constant coefficients of the series.

Remarks:

- For each fixed x-value, the series

$$\sum_{n=0}^{\infty} c_n x^n$$

is a series of constants that we can test for convergence or divergence

- A power series may converge for some values of x and may diverge for other values of x.

- The sum of the series is a function

like a polynomial except it has infinite # of terms

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

whose domain is the set of all $x \in \mathbb{R}$ for which

$$\sum_{n=1}^{\infty} c_n x^n \text{ converges.}$$

Example 8.5.0 p. 593

Consider the power series $\sum_{n=0}^{\infty} x^n$ (where $C_n = 1$).

To find the x-values for which this series converges,

we refer back to our derivation of results for the geometric serie

$$\sum_{n=0}^{\infty} x^n = \lim_{N\to\infty} \sum_{n=0}^{N} x^n$$

$$= \lim_{N\to\infty} \frac{1-x^{N+1}}{1-x}$$

We see this limit exists and is finite iff $|x| < 1 \Rightarrow -1 < x < 1$

$\Rightarrow$ The power series $\sum_{n=0}^{\infty} x^n$ $\begin{cases} \text{converges if} & -1 < x < 1 \\ \text{diverges if} & |x| \geq 1 \end{cases}$

Example 8.5.1 p. 593 A Power Series that converges at only one point

For what $x \in \mathbb{R}$ does the series

$$\sum_{n=0}^{\infty} n!\, x^n$$

converge?

Solution: We see the sequence of terms $a_n = n!\, x^n$ involves both a factorial and a power. Thus, we'll try the ratio test.

To this end consider (when $x \neq 0$)

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right|$$

$$= \lim_{n\to\infty} (n+1)\,|x| \to \infty$$

Then, by the ratio test $\sum_{n=0}^{\infty} n!\, x^n$ is divergent for $x \neq 0$.

On the other hand, if $x = 0$,

$$\sum_{n=1}^{\infty} n!\, x^n = \sum_{n=1}^{\infty} 0 = 0 \quad \text{converges} \checkmark$$

Thus, the given series converges only for $x = 0$. $\square$

Example 8.5.2 p. 593 For what values ~~does~~ of x does the series

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} \text{ converge?}$$

Solution : Consider the series given in this problem.

Let $a_n = \dfrac{(x-3)^n}{n}$.

□ What test might we use here?

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$

$$= \frac{n}{n+1} \cdot |x-3|$$

$$= \frac{1}{1+\frac{1}{n}} \cdot |x-3|$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \cdot |x-3|$$

$$= |x-3|$$

By the ratio test, we know our series is ~~convergent~~ absolutely convergent (and thus convergent) iff $|x-3| < 1$

$$\Rightarrow -1 < x-3 < 1$$

$$\Rightarrow 2 < x < 4$$

- We also know by the ratio test that our series diverges when $|x-3|>1$

$$\Rightarrow \quad x-3>1 \quad \text{or} \quad x-3<-1$$

$$\Rightarrow \quad x>4 \quad \text{or} \quad x<2$$

- we recall that the ratio test is inconclusive (gives no information) when $|x-3|=1$. Then, we consider $x=2$ and $x=4$ separately.

Case $x=4$: $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=0}^{\infty} \frac{(4-3)^n}{n} = \sum_{n=0}^{\infty} \frac{1}{n}$ divergent

Case $x=2$: $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{n} = \sum_{n=0}^{\infty} \frac{(2-3)^n}{n}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \quad \text{convergent.}$$

Thus, the given series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$ converges for $2 \le x < 4$.

Definition 8.5.2 p. 593 A series in the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called
- a power series in $(x-a)$
- a power series centered at a
- a power ~~seriees~~ series about a

Remarks:
- Writing out the term $n=0$, we adopt the convention that $(x-a)^0 = 1$ even when $x=a$.
- When $x=a$, all of the terms are 0 for $n \geq 1$, so the power series always converges when $x=a$.

Major idea: Power series provide an alternate (and useful) description of functions that arise in mathematics, chemistry & physics.

Example 8.5.3 p.594 A Bessel Function (Related to HW problem 8.5.28, 8.5.29, 8.5.30)

Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Solution: Let $a_n = \dfrac{(-1)^n \cdot x^{2n}}{2^{2n}\cdot(n!)^2}$ be the sequence of terms.

Consider

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}\, x^{2(n+1)}}{2^{2(n+1)}\cdot[(n+1)!]^2} \cdot \frac{2^{2n}\cdot(n!)^2}{(-1)^n\, x^{2n}}\right|$$

$$= \left|\frac{(-1)^n \cdot x^{2n+2} \cdot 2^{2n}\cdot(n!)^2}{x^{2n}\cdot 2^{2n+2}\,(n!\,(n+1))^2}\right|$$

$$= \frac{x^2}{4(n+1)^2}$$

Then, for any $x \in \mathbb{R}$, we see

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0 < 1$$

$\Rightarrow \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{2^{2n}(n!)^2}$ converges for all $x \in \mathbb{R}$

$\Rightarrow$ The domain of $J_0(x)$ is $\mathbb{R} = (-\infty, \infty)$

Written Hw Hints:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(-1)^n \cdot x^{2n}}{2^{2n}\cdot(n!)^2}$$

$= \lim_{N\to\infty} S_N$ where $S_N = \displaystyle\sum_{n=0}^{N} \frac{(-1)^n \cdot x^{2n}}{2^{2n}(n!)^2}$

We can graph the first few terms of this sequence.

$$S_0(x) = \frac{(-1)^0 \ x^0}{2^0 \cdot (0!)^2} = \frac{1 \cdot 1}{1 \cdot 1} = 1 \ \checkmark$$

$$S_1(x) = \sum_{n=0}^{1} \frac{(-1)^n \ x^{2n}}{2^n \cdot (n!)^2}$$

$$= 1 + \frac{(-1)^1 \cdot x^2}{2^2 \cdot (1!)^2} =$$

$$= 1 - \frac{x^2}{4} \ \checkmark$$

$$S_2(x) = \sum_{n=0}^{2} \frac{(-1)^n \cdot x^{2n}}{2^{2n} \cdot (n!)^2}$$

$$= 1 - \frac{x^2}{4} + \frac{(-1)^2 \cdot x^4}{2^4 \cdot (2!)^2}$$

$$= 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

$$S_3(x) = \sum_{n=0}^{3} \frac{(-1)^n \ x^{2n}}{2^{2n} (n!)^2}$$

$$= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304}$$

Theorem 8.5.3 p. 595

For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are only three possibilities

i. The series converges only for $x=a$

ii. The series converges for all $x \in \mathbb{R}$.

Special case → iii. There is a positive number R such that the series converges if

$$|x-a| < R$$

and diverges if

$$|x-a| > R$$

- The number R in case (iii) is called the **radius of convergence** of the power series.
  - in case i, $R=0$ (by convention)
  - in case ii, $R=\infty$

- The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.

Interval of Convergence

Case (i) : $R = 0 \Rightarrow$ The interval of convergence is a single point a

Case (ii) : $R = \infty \Rightarrow$ The inverval of convergence is all of $\mathbb{R}$

Case (iii) : $R > 0$ & $R < \infty \Rightarrow$

$$|x-a| < R \Rightarrow -R < x-a < R$$

$$\Rightarrow a-R < x < a+R$$

$\Rightarrow$ Endpoints $x = a \pm R$ are wildcards (anything can happen)

$\Rightarrow$ There are four possibilities for the interval of convergence

$$(a-R, a-R)$$

$$(a-R, a+R]$$

$$[a-R, a+R)$$

$$[a-R, a+R]$$

Let's summarize the radius of convergence and interval of convergence for each example so far

Series	Radius of Convergence	Interval of Convergence
$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
$\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = +\infty$	$(-\infty, \infty)$

- The Ratio Test can be ~~useets~~ used to determine the radius of convergence R in most cases.

The Ratio Test fails when x is an endpoint of the interval of convergences, so endpoints must be checked w/ other tests.

Example 8.5.4 p. 596   Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

Solution: Let $a_n = \dfrac{(-3)^n \cdot x^n}{\sqrt{n+1}}$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1} \, x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n \cdot x^n}\right|$$

$$= \left| -3x \cdot \frac{\sqrt{n+1}}{\sqrt{n+2}} \right|$$

$$= 3 \cdot |x| \cdot \frac{\sqrt{1 + 1/n}}{\sqrt{1 + 2/n}}$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3|x|$$

$\Rightarrow$ Series converges if $3 \cdot |x| < 1$

$$\Rightarrow -\frac{1}{3} < x < \frac{1}{3}$$

$$\Rightarrow R = 1/3$$

Now we test end points:

Case $x = 1/3 \Rightarrow \sum_{n=0}^{\infty} \frac{(-3)^n \cdot \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converges by Alt Series Test

Case $x = -1/3 \Rightarrow \sum_{n=0}^{\infty} \frac{(-3)^n \cdot \left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ diverges by p-series test

Then we see the interval of convergence is

$$\left\{x \in \mathbb{R} : -\frac{1}{3} < x \leq \frac{1}{3}\right\} = \left(-\frac{1}{3}, \frac{1}{3}\right]$$

Example 8.5.5 p.596-597

Find the Radius of Convergence and interval of convergence for series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

Solution: Let $a_n = \dfrac{n \cdot (x+2)^n}{3^{n+1}}$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)\cdot(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n\cdot(x+2)^n}\right|$$

$$= \left|\frac{n+1}{n} \cdot \frac{(x+2)}{3}\right|$$

$$= \left(1+\frac{1}{n}\right) \cdot \frac{|x+2|}{3}$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+2|}{3}$$

$\Rightarrow$ series converges if $\dfrac{|x+2|}{3} < 1$ by ratio test

$\Rightarrow |x+2| < 3$ $(R=3)$

$\Rightarrow -3 < x+2 < 3$

$\Rightarrow \quad -5 < x < 1$

$\Rightarrow \quad R = 3$ and we want to check endpoints

Case $x = -5$: $\displaystyle\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} n \cdot (-1)^n$ diverges by test for divergence

Case $x = 1$: $\displaystyle\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} n$ diverges by test for divergence

$\Rightarrow$ Interval of convegence is $(-5, 1)$