The origin of the word

digit is from the Latin

word digitus meaning

finger or toe and arose

from the practice of counting

on the fingers

The number of digits in a numeric value is the total number of digits used to express the entire written value in a decimal representation:

eg$_1$: 123.4567

1st digit | 2nd digit | 3rd digit | 4th digit | 5th digit | 6th digit | 7th digit

this number is a 7-digit number
Since it is written with 7 digits

Let's look at another example:

eg. The number below is given as a 11-digit number

$$eg_2: \quad \underset{\uparrow\ \text{1st digit}}{2}\,.\,\underset{\uparrow\ \text{2nd digit}}{7}\,\underset{\uparrow\ \text{3rd digit}}{1}\,\underset{\uparrow\ \text{4th digit}}{8}\,2\,8\,1\,8\,2\,8\,\underset{\uparrow\ \text{11th digit}}{5}$$

One of the immediate questions that arises is what do these digits mean?

To answer this question, let's look at some examples.

Let's take a look at an 8-digit number

decimal point

eg3: 306.01968

number of $10^2 = 100$

number of $10^1 = 10$

number of $10^0 = 1$

number of $10^{-1} = 0.1$

number of $10^{-2} = 0.01$

number of $10^{-3} = 0.001$

number of $10^{-4} = 0.0001$

number of $10^{-5} = 0.00001$

integer part of a number

fractional part of number

seperated by decimal point

We can write this number out as a finite sum:

$$306.01968$$

$$= \underset{\text{hundreds}: 10^2}{300} + \underset{\text{tens}: 10^1}{0} + \underset{\text{ones}: 10^0}{6} + \underset{\text{tenths}: 10^{-1}}{0.0} + \underset{\text{hundredth}: 10^{-2}}{0.01} + \underset{\text{thousandths}: 10^{-3}}{0.009} + \underset{10^{-4}}{0.0006} + \underset{10^{-5}}{0.0000}$$

$$= 3 \cdot 10^2 + 0 \cdot 10^1 + 6 \cdot 10^0 + 0 \cdot 10^{-1} + 1 \cdot 10^{-2} + 9 \cdot 10^{-3} + 6 \cdot 10^{-4} + 8 \cdot 10$$

$$= d_2 \cdot 10^2 + d_1 \cdot 10^1 + d_0 \cdot 10^0 + d_{-1} \cdot 10^{-1} + \cdots + d_{-5} \cdot 10^{-5}$$

where we can define the <u>sequence</u>

$$d_2 = 3, \; d_1 = 0, \; d_0 = 6, \; d_{-1} = 0, \; d_{-2} = 1$$

eg 4: Let's look at another example:

decimal point

$$3.\underbrace{1416}_{} = 3 + 0.1 + 0.04 + 0.001 + 0.0006$$

integer part of number

fractional part of number

$$= 3 \cdot 1 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{6}{10{,}000}$$

$$= 3 \cdot 10^{0} + 1 \cdot 10^{-1} + 4 \cdot 10^{-2} + 1 \cdot 10^{-3} + 6 \cdot 10^{-4}$$

$$= d_0 \cdot 10^{0} + d_{-1} \cdot 10^{-1} + d_{-2} \cdot 10^{-2} + d_{-3} \cdot 10^{-3} + d_{-4} \cdot 10^{-}$$

where $d_0 = 3$

$d_{-1} = 1$

$d_{-2} = 4$

$d_{-3} = 1$

$d_{-4} = 6$

These examples have a pattern: for $y \in \mathbb{R}$, we can write the general decimal representation of y as

$$y = \cdots \underbrace{d_3\, d_2\, d_1\, d_0}_{\text{integer part of } y} \overset{\substack{\text{decimal point} \\ \downarrow}}{.} \underbrace{d_{-1}\, d_{-2}\, d_{-3}}_{\text{fractional part of } y} \cdots$$

$$= \cdots + d_3 \cdot 10^3 + d_2 \cdot 10^2 + d_1 \cdot 10^1 + d_0 \cdot 10^0 \underset{\substack{\uparrow \\ \text{decimal} \\ \text{separator} \\ \text{marks} \\ \text{here}}}{+} d_{-1} \cdot 10^{-1} + \cdots$$

We say that a nonnegative rational number $y \in \mathbb{Q}$ can be written exactly using a unique finite n-digit decimal representation iff

$$y = \underbrace{d_i \, d_{i-1} \cdots d_1 \, d_0}_{\text{the integer part of } y} \underset{\substack{\uparrow \\ \text{decimal point}}}{\bullet} \underbrace{d_{-1} \, d_{-2} \cdots d_{-f}}_{\text{the decimal part of } y}$$

where A. $d_k \in \{0, 1, 2, \ldots, 9\}$ for all $k \in \{-f, -f+1, \ldots, -2, -1, 0, 1, \ldots i\}$

B. $d_i \neq 0$

C. $d_{-f} \neq 0$

where $i \in \mathbb{Z}_0^+$ and $f \in \mathbb{N}$ with $n = i + f + 1$.

For any rational number $y \in \mathbb{Q}$ that can be written exactly using a finite decimal representation, we can adopt conventions to give a unique, n-digit decimal representation

$\boxed{\text{Example 5}}$

$$\underbrace{002.71828}_{\text{this number is written with 8 digits}} = \underbrace{2.71828}_{\text{this number is written with 6 digits}}$$

egs: $\overbrace{002}^{\text{integer part}}\overset{\text{decimal point}}{.}\overbrace{71828}^{\text{fractional part}}$

these two digits give us no useful information about this number since the integer part of this number is 2:

$$002 = 02 = 2 = 0002 \ldots$$

In other words, when writing a rational number in decimal form, let's agree NOT to write "leading" zeros on the integer part of our number.

Convention 1: The first digit of any $y \geq 1$ should always be non zero

For a positive rational number $y \in \mathbb{Q}$

with an exact finite decimal representation

$$y = d_i d_{i-1} \cdots d_1 d_0 . d_{-1} d_{-2} \cdots d_{-f}$$

let us assume that as long as

the integer part of our number

is NOT zero, (i.e. $y \geq 1$) with

$$\text{int}(y) = d_i d_{i-1} \cdots d_1 d_0 \neq 0$$

then we will require $d_i \neq 0$

eg 5 continued...:

Using convention 1, let's

rewrite the number below

$$y = \underbrace{0020}_{\text{integer part}}\underbrace{.}_{\text{decimal point}}\underbrace{00147}_{\text{fractional part}} = \overbrace{20.00147}^{\text{7-digit number}}$$

$$= \underbrace{\overset{i=1}{d_1} d_0 . d_{-1} d_{-2} d_{-3} d_{-4} \overset{f=}{d}}$$

this is an n-digit representation where

$$n = i + 1 + f$$
$$= 1 + 1 + 5 = 7$$

Convention 1 : Since ~~y>1~~, the first digit written in the integer part should be non zero

$$\text{int}(y) = \boxed{00}20 = 20$$

not necessary for our purpose

Next, let's develop a similar convention for the decimal part of a rational number. Suppose $y \in \mathbb{Q}$ is a positive number with a finite exact decimal representation

$\boxed{\text{Example 6}}$

$$y = \underbrace{1.732050800000}_{\text{this number is written with 13 digits}}$$

$$\Rightarrow y = \underbrace{1.7320508}_{\text{this number is written with 8 digits}}$$

integer part

decimal point

$$1.\overbrace{7320508\underbrace{00000}_{\text{these five digits give us no useful information about this number}}}^{\text{fractional part}}$$

In other words, when writing a positive rational number that has an exact finite decimal representation, let's agree NOT to write "trailing" zeros on the integer part of our number

Convention 2: The last digits of $y > 0$ should be nonzero whenever possib[le]

For a positive rational number $y \in \mathbb{Q}$ with an exact finite decimal representation

$$y = d_i \, d_{i-1} \cdots d_1 \, d_0 \, . \, d_{-1} \, d_{-2} \cdots d_{-f}$$

let us assume that as long as no rounding is involved, then the last digit of the fractional part of the number is NOT zero:

$$\text{frac}(y) = d_{-1} \, d_{-2} \cdots d_{-f}$$

with $d_{-f} \neq 0$

Using convention 2, let's rewrite our number

$$y = \underbrace{1.732050800000}_{\text{non unique 13-digit decimal representation}}$$

$$= \underbrace{1.7320508}_{\text{unique, 8-digit, exact representation of } y}$$

convention 1; convention 2

$$= \underbrace{\overset{i=0}{d_0} . d_{-1} d_{-2} d_{-3} d_{-4} d_{-5} d_{-6} \overset{f=7}{d_{-7}}}_{\text{this is an n-digit representation}}$$

with $n = i + 1 + f$

$\Rightarrow n = 0 + 1 + 7 = 8$.

Let's move on to considering rational numbers $y \in \mathbb{Q}$ with finite decimal representations where $0 < y < 1$.

Example 7

$$y = \underbrace{00.000264510}_{\substack{\text{this number is written} \\ \text{with 11-digits}}}$$

Notice, something special is going on in the integer part as well as the first few digits of the decimal part of this number.

decimal point

$$\overbrace{00}^{\text{integer part}}.\overbrace{0000264510}^{\text{fractional part}}$$

not useful by convention 1

not useful by convention

these four digits <u>DO</u> give us useful information: these "leading" zeros help to identify where the decimal point should be.

In other words, when writing an exact finite representation of $y \in \mathbb{Q}$ with $0 < y < 1$, let's agree to write a single zero in integer part and all leading zeros in fractional part.

Convention 3: The first digit of $0<y<1$ should be a single zero in integer part followed by fractional part as per convention 2

For a positive rational number $y \in \mathbb{Q}$ where $0<y<1$ that has a unique, exact, finite decimal representation

$$y = d_0 . d_{-1} d_{-2} \dots d_{-f}$$

let us agree to write $d_0 = 0$ and $d_{-f} \neq 0$.
(convention 2)

Using convention 3, let's rewrite our number:

$$y = \underbrace{00.000264510}_{\text{non unique 11-digit decimal representation}}$$

convention 3 ↓ ... ↙ convention 2

$$= \underbrace{0.00026451}_{\text{unique 9-digit, exact decimal representation}}$$

$$= \underbrace{\overset{i=0}{d_0} . d_{-1} d_{-2} d_{-3} d_{-4} d_{-5} d_{-6} d_{-7} \overset{f=8}{d_{-8}}}_{\text{this is an n-digit representation with } n = i+1+f}$$

$\Rightarrow n = 0 + 1 + 8 = 9$

Example 6

convention 1 convention 2

$$y = \underbrace{1.7320508}_{\text{8 digit decimal representation}}$$

conventions 1 & 2

$$\Rightarrow \quad \mathrm{int}(y) = 1$$

convention 2

$$\mathrm{frac}(y) = 0.\underbrace{7320508}_{\text{7 decimal digit}}$$

convention 3

Example 7

$$y = \underbrace{\overset{\text{convention 3}}{0}.0002645\overset{\text{convention 2}}{1}}_{\text{9-digit representation}}$$

$$\Rightarrow \quad \text{int}(y) = 0$$

$$\text{frac}(y) = \underset{\text{convention 3}}{0}.\underbrace{0002645\overset{\text{convention 2}}{1}}_{\text{8-decimal digits}}$$

The <u>number of decimal digits</u> in a numeric value is the total number of digits (including zeros) to the right of the decimal point in the unique decimal representation (that conforms to our conventions 1,2,3):

convention 1

convention 2

$\boxed{\text{Example 5}}$ $y = \underbrace{2.71828}_{\text{6-digit representation}}$

conventions 1 & 2

$$\Rightarrow \quad \mathrm{int}(y) = 2$$

convention

$$\mathrm{frac}(y) = 0.\underbrace{71828}_{\text{5 decimal digits}}$$

convention 3

Lesson 0: Finite decimal approximations

Recall from the last video that any real number $y \in \mathbb{R}$ with $y \geq 0$ can be represented exactly

$$y = \sum_{k=0}^{\infty} \frac{a_k}{10^k}$$

← infinite decimal representation

$$\{a_k\}_{k=0}^{\infty} = \{a_0, a_1, a_2, \dots\}$$

sequence of digits

$$y = \sum_{k=0}^{\infty} \frac{a_k}{10^k}$$

Since the upper limit of this sum is infinite, this gives the exact value of $y \in \mathbb{R}$

You may recall that when working with y to solve a problem, we may want to perform a calculation in which we don't need y exactly...

Instead, we might only need to approximate y.

You may also recall that the exact value of $y \in \mathbb{R}$ can be approximated to any "accuracy" with a finite decimal representation

$$y = a_0 . a_1 a_2 a_3 a_4 \ldots$$

$$\Rightarrow y = \sum_{k=0}^{\infty} \frac{a_k}{10^k}$$

exact valve via infinite series

finite approximat

$$\Rightarrow y \approx a_0 . a_1 a_2 a_3 \ldots a_n = \sum_{k=0}^{n} \frac{a_k}{10^k}$$

Let's call the number

$$\hat{y} = \sum_{k=0}^{n} \frac{a_k}{10^k} = a_0 . a_1 a_2 \cdots a_n$$

an approximation to the real number y.

Let's look at some famous examples

infinite

$$\underbrace{\pi}_{\substack{\text{"explicit"} \\ \text{representation}}} = \underbrace{\sum_{k=0}^{\infty} \frac{a_k}{10^k}}_{\substack{\text{exact} \\ \text{value}}} \quad \text{w/} \quad \begin{aligned} a_0 &= 3 \\ a_1 &= 1 \\ a_2 &= 4 \\ a_3 &= 1 \\ a_4 &= 5 \\ a_5 &= 9 \\ &\vdots \end{aligned}$$

approximation

$$\hat{\pi} = 3.1416 \approx \pi \leftarrow \text{exact value}$$

The absolute error between these is

$$|\pi - \hat{\pi}| = |3.14159265\ldots - 3.1416|$$

$$= |-0.0000007346\ldots|$$

$$= 0.000007346\ldots$$

Another famous example is

$$y = \sqrt{2} = \underbrace{\sum_{k=0}^{\infty} \frac{a_k}{10^k}}_{\text{exact value}}$$

"explicit" representation

$a_0 = 1$
$a_1 = 4$
$a_2 = 1$
$a_3 = 4$
$a_4 = 2$
$a_5 = 1$
$a_6 = 3$
$\vdots$

We can approximate this value

$$\hat{y} = 1.41421 \approx \sqrt{2} = y$$

approximation

exact value

Again, we can quantify the "absolute error" between these as

$$|y - \hat{y}| = |1.41421356\ldots - 1.41421|$$

$$= |0.00000356\ldots|$$

$$= 0.00000356\ldots$$

$$y = \underbrace{3}_{\text{integer part}}\underset{\uparrow\ \text{decimal point}}{.}\underbrace{14159}_{\text{fractional part}}$$

$$\Rightarrow \quad y = 3.14159$$

$$\Rightarrow \quad y = 3 + 0.1 + 0.04 + 0.001 + 0.0005 + 0.0000$$

$$\Rightarrow y = \frac{3}{10^0} + \frac{1}{10^1} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5}$$

$$\Rightarrow y = \frac{a_0}{10^0} + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \frac{a_4}{10^4} + \frac{a_5}{10^5}$$

where $a_0 = 3$, $\leftarrow$ integer part

$$\left.\begin{aligned} a_1 &= 1 \\ a_2 &= 4 \\ a_3 &= 1 \\ a_4 &= 5 \\ a_5 &= 9 \end{aligned}\right\} \leftarrow \text{fractional part}$$

$$\Rightarrow y = \sum_{k=0}^{5} \frac{a_k}{10^k}$$

this is called a finite series

$$y = \underbrace{20}_{\text{integer part}}\underset{\text{decimal point}}{.}\underbrace{085537}_{\text{fractional part}}$$

$$\Rightarrow\ y = \frac{20}{10^0} + \frac{0}{10^1} + \frac{8}{10^2} + \frac{5}{10^3} + \frac{5}{10^4} + \frac{3}{10^5} + \frac{7}{10^6}$$

$$\Rightarrow y = \frac{a_0}{10^0} + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \frac{a_4}{10^4} + \frac{a_5}{10^5} + \frac{a_6}{10^6}$$

where $a_0 = 20$ ⟵ integer part

$$\left.\begin{aligned} a_1 &= 0 \\ a_2 &= 8 \\ a_3 &= 5 \\ a_4 &= 5 \\ a_5 &= 3 \\ a_6 &= 7 \end{aligned}\right\} \longleftarrow \text{fractional part}$$

$$\Rightarrow y = \sum_{k=0}^{6} \frac{a_k}{10^k}$$

$$y = \underbrace{306}_{\text{integer part}}\underset{\uparrow\ \text{decimal point}}{.}\underbrace{0196848}_{\text{fractional part}}$$

Then, we can write

$$y = \frac{306}{10^0} + \frac{0}{10^1} + \frac{1}{10^2} + \frac{9}{10^3} + \frac{6}{10^4} + \frac{8}{10^5} + \frac{4}{10^6} + \frac{8}{10^7}$$

$$y = \frac{a_0}{10^0} + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \frac{a_4}{10^4} + \frac{a_5}{10^5} + \frac{a_6}{10^6} + \frac{a_7}{10^7}$$

where $a_0 = 306$ ← integer part

$$\left.\begin{aligned} a_1 &= 0 \\ a_2 &= 1 \\ a_3 &= 9 \\ a_4 &= 6 \\ a_5 &= 8 \\ a_6 &= 4 \\ a_7 &= 8 \end{aligned}\right\}$$ ← fractional part

$$y = \sum_{k=0}^{7} \frac{a_k}{10^k}$$

Notice we can write the decimal representation of a nonnegative real number $y \in \mathbb{R}$ in the form

$$y = \underbrace{a_0}_{\text{the integer part of } y}\underset{\uparrow \text{ decimal point}}{\bullet}\underbrace{a_1\, a_2\, a_3\, a_4\, a_5\, a_6 \ldots}_{\text{the fractional part of } y}$$

$$\Rightarrow y = \frac{a_0}{10^0} + \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \frac{a_4}{10^4} + \frac{a_5}{10^5} + \cdots$$

$$\Rightarrow y = \sum_{i=0}^{\infty} \frac{a_i}{10^i}$$

Notice in this decimal representation

$$y = \sum_{k=0}^{\infty} \frac{a_k}{10^k}$$

$$= \underbrace{a_0}_{\uparrow \text{ integer part}} \underset{\uparrow \text{ decimal point}}{.} \underbrace{a_1 a_2 a_3 a_4 a_5 \cdots}_{\uparrow \text{ fractional part}}$$

- a_0 is a nonnegative integer
- $a_1, a_2, a_3, \ldots$ are integers with $0 \le a_k < 9$ for all $k \in \mathbb{N}$