3.3 Matrices from Outer Products

Definition 3.16: Outer Product of Vectors

Let $\mathbf{x} \in \mathbb{R}^{m\times 1}$ and $\mathbf{y} \in \mathbb{R}^{n\times 1}$. Then, the outer product between $\mathbf{x}$ and $\mathbf{y}$ is the $m \times n$ matrix given by

$$\mathbf{x}\mathbf{y}^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix}$$

Notice that row i of the outer product $\mathbf{x}\mathbf{y}^T$ is given by $x_i\mathbf{y}^T$ for $i = 1, 2, ..., m$ while column k of the outer product is given by $y_k\mathbf{x}$ for $k = 1, 2, ..., n$.

The outer product between two vectors can be very useful in computing many multiplications between real numbers simultaneously. These matrices are also extremely helpful in generating new matrices.

EXAMPLE 3.3.1

Let $\{\mathbf{e}_i\}_{i=1}^n \subseteq \mathbb{R}^{n\times 1}$ be the set of *elementary basis vectors* for $\mathbb{R}^n$. We will define each vector $\mathbf{e}_i$ component-wise as follows

$$\mathbf{e}_i(j,1) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 3$, we see

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Using these vectors and the outer product, we can create $n \times n$ *matrix units*

$$E_{ik} = \mathbf{e}_i\mathbf{e}_k^T$$

with all zero entries except the entry in row i and column k, which is equal to 1. In the 3×3 case, there are a total of nine matrix units given as follows:

$$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{31} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad E_{32} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad E_{33} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice, we can actually define matrix units in the $m \times n$ case, although to do so we would need to take an outer product between vectors of different sizes. For now we focus on the square matrix units as we will use these most frequently.

EXAMPLE 3.3.2

Lets suppose we want to create a 5×5 matrix T having the following structure:

$$T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{bmatrix}$$

Then, we can define a 5×1 vector

$$\boldsymbol{\tau} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ -2 \\ 4 \end{bmatrix}$$

and we can write $T = \boldsymbol{\tau}\, \mathbf{e}_2^T$, where $\mathbf{e}_2$ is the second elementary basis vector in $\mathbb{R}^5$.

Lesson 9, Part 1: Matrices from Outer Products- Suggested Problems

For all the problems below, be sure to explicitly state the dimensions of the matrices you use for each model.

1. Create each of the following matrices using an outer product between two vectors. Specifically state the two vectors you are using to write the outer product:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -3 & 0 \\ 0 & 2 & 0 \end{bmatrix},$$

3.4 Matrix Addition, Scalar Multiplication and the Transpose

Now that we have seen the use of matrices to model a selection of important applications, let's discuss operations that exist between matrices. We begin with matrix-matrix addition and scalar-matrix multiplication. As we will see, these are the natural analogs of the corresponding vector operations with the same name.

Definition 3.17: Matrix-Matrix Addition

If $A, B \in \mathbb{R}^{m\times n}$ are matrices with the same row and column dimensions, then the **matrix sum** $A + B$ is the matrix obtained by adding the corresponding entries of A and B. In other words, the entry with row index i and column index k in the sum $A + B$ is the sum $a_{ik} + b_{ik}$.

EXAMPLE 3.4.1

In Section 3.1 we defined the $n \times n$ identify matrix I_n as the matrix containing all ones on the main diagonals. We can use matrix addition to write the identity as the sum of n matrix units as follows:

$$I_n = \sum_{i=1}^{n} E_{ii} = \sum_{i=1}^{n} \mathbf{e}_i\, \mathbf{e}_i^T$$

Definition 3.18: Scalar-Matrix Multiplication

Suppose $A \in \mathbb{R}^{m\times n}$ is a matrix and $\alpha \in \mathbb{R}$ is a scalar. Then the **scalar-matrix product** $\alpha \cdot A$ is the matrix obtained by multiplying each entries of A by the scalar α. In other words, the entry with row index i and column index k in the product $\alpha \cdot A$ is the product $\alpha\, a_{ik}$.

EXAMPLE 3.4.2

We can use matrix-matrix addition and scalar-matrix multiplication to write any square matrix as the sum of matrix units. For $A \in \mathbb{R}^{n\times n}$, we have

$$A = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} \cdot \mathbf{e}_i\, \mathbf{e}_k^T$$

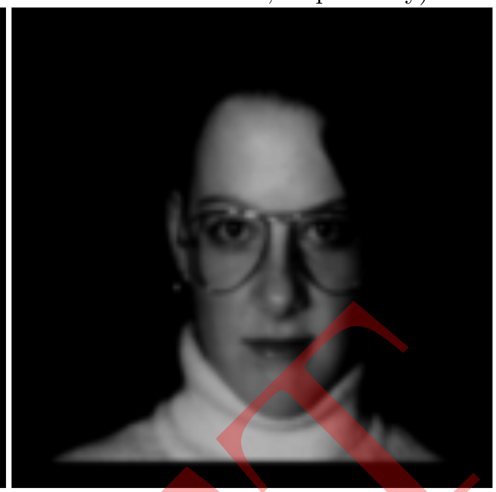
For example, in the $n = 2$ case, we might write

$$\begin{bmatrix} 2 & -1 \\ -3 & 5 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + -1 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + -3 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 5 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 2 \cdot \mathbf{e}_1\, \mathbf{e}_1^T + -1 \cdot \mathbf{e}_1\, \mathbf{e}_2^T + -3 \cdot \mathbf{e}_2\, \mathbf{e}_1^T + 5 \cdot \mathbf{e}_2\, \mathbf{e}_2^T$$

EXAMPLE 3.4.3

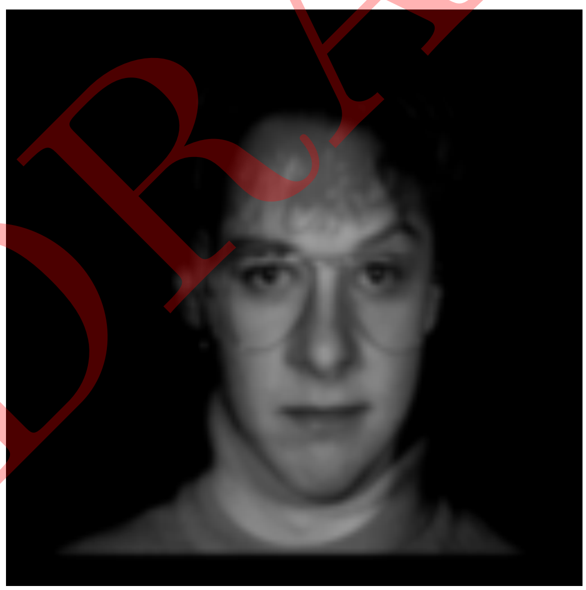
Let's consider two image vectors that compose faces from the Face Recognition Project at MIT (http://courses.media.mit.edu/2004fall/mas622j/04.projects/faces/). Each images is stored as a 128×128 black-and white photo. Here we take two images from our RAW image data base (image numbers 2147 and 1734, respectively):



We will store the the image on the left in the matrix M and the image on the right in the matrix F. Notice that $M, F \in \mathbb{R}^{128\times128.}$ Now, we can use matrix operations to blend these two images together. In particular, let's consider the new 128×128 blended matrix

$$B = \frac{1}{2}M + \frac{1}{2}F$$

This blended image matrix B is shown below:



This is a great example of how meaning can be introduced for the matrix-matrix addition operation.

There are a number of algebraic properties that hold for these operations between matrices.

Theorem 16: Algebraic Properties of Matrix Operations

Let $A, B, C \in \mathbb{R}^{m\times n}$ and $a, b \in \mathbb{R}$. Then, all of the following are properties of matrix addition:

1. Commutativity of matrix addition: $A + B = B + A$
2. Associativity of matrix: $A + (B + C) = (A + B) + C$
3. Additive Identity: $A + 0 = 0 + A = A$
4. Additive Inverses: $A + -A = -A + A = 0$
5. Distributivity of matrix addition: $a(A + B) = aA + aB$
6. Distributivity of scalar addition: $(a + b)A = aA + bA$
7. Associativity of scalar multiplication: $a\,(bA) = (ab)A$
8. Multiplicative Identity of scalar multiplication: $1A = A$

Notice that the properties of matrix addition and scalar-matrix multiplication are identical to the corresponding properties of vector addition and scalar-vector multiplication.

Proof. Let $A, B, C \in \mathbb{R}^{m\times n}$ and $a, b \in \mathbb{R}$. We begin by establishing commutativity of matrix addition. Consider

$$\begin{aligned} A + B &= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \\ &= \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} & \cdots & b_{1n}+a_{1n} \\ b_{21}+a_{21} & b_{22}+a_{22} & \cdots & b_{2n}+a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}+a_{m1} & b_{m2}+a_{m2} & \cdots & b_{mn}+a_{mn} \end{bmatrix} \\ &= B + A \end{aligned}$$

In this case, we've used the scalar properties of $\mathbb{R}$ and the component-wise definition of matrix addition to confirm commutativity.

Let's also confirm associativity. Again, we begin by considering

$$A+(B+C)=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} & \cdots & b_{1n}+c_{1n} \\ b_{21}+c_{21} & b_{22}+c_{22} & \cdots & b_{2n}+c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1}+c_{m1} & b_{m2}+c_{m2} & \cdots & b_{mn}+c_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} & \cdots & a_{1n}+b_{1n}+c_{1n} \\ a_{21}+b_{21}+c_{21} & a_{22}+b_{22}+c_{22} & \cdots & a_{2n}+b_{2n}+c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1}+c_{m1} & a_{m2}+b_{m2}+c_{m2} & \cdots & a_{mn}+b_{mn}+c_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

$$= (A+B)+C$$

Again, we've borrowed the associativity of each coefficient from $\mathbb{R}$. The other proofs from this theorem follow from similar arguments and are left to the reader as exercises. □

Definition 3.19: Rank-one updates

Let $A \in \mathbb{R}^{m\times n}$ and vectors $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$. Then, a *rank one update of A* is given by

$$A + \mathbf{x}\,\mathbf{y}^T$$

As we will see, rank one modifications of matrices play an important role in our solutions for the linear-systems problem, the least-squares problem and the eigenvalue problems. There are many important examples of rank one updates to the identity matrix. Below are a few major classes of rank one updates that we will revisit in our discussion of matrix-vector multiplication and linear systems problems.

Definition 3.20: Shear Matrices

Let $n \in \mathbb{N}$. For $i \neq k$, we define an $n \times n$ *shear matrix* takes the form

$$S_{ik}(c) = I_n + c \cdot \mathbf{e}_i \, \mathbf{e}_k^T$$

Notice that shear matrices are rank one updates to the identity matrix. Also, shear matrices are the identity matrix with entry in row i and column k equal to c.

EXAMPLE 3.4.4

Let's look at $S_{31}(-5)$ in the $n = 3$ case. Notice

$$\begin{aligned} S_{31}(-5) &= I_3 + -5 \cdot \mathbf{e}_3 \, \mathbf{e}_1^T \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + -5 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -5 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \end{aligned}$$

Definition 3.21: Dialation Matrices

Let $n \in \mathbb{N}$. For $j \in \{1, 2, ..., n\}$, we define an $n \times n$ *dilation matrix*

$$D_j(c) = I_n + (c-1) \cdot \mathbf{e}_j \, \mathbf{e}_j^T$$

EXAMPLE 3.4.5

For the $n = 5$ case, we can look at

$$\begin{aligned}
D_2(5) &= I_5 + (5-1) \cdot \mathbf{e}_2 \, \mathbf{e}_2^T \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + (5-1) \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 5-1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

As we will see, we can use this dilation matrix to scale an entire row or an entire column of a matrix via matrix-matrix multiplication.

Definition 3.22: Transposition Matrices

Let $n \in \mathbb{N}$. For $i \neq k$, we define the *transposition matrix* in $\mathbb{R}^{n\times n}$ to be given by

$$P_{ik} = \mathbf{e}_i\,\mathbf{e}_k^T + \mathbf{e}_k\,\mathbf{e}_i^T + \sum_{\substack{j=1\\ j\neq i,k}}^{n} \mathbf{e}_j\,\mathbf{e}_j^T$$

with $\mathbf{e}_i \in \mathbb{R}^n$ for all $i = 1, 2, ..., n$.

Notice, we can form any transposition matrix by taking the $n \times n$ identity matrix and swapping row i with row k. Transposition matrices are an example of what is know as a rank-two update.

EXAMPLE 3.4.6

Let's consider the transposition matrix P_{24} in $R^{4\times 4}$ given by

$$P_{24} = \mathbf{e}_2\,\mathbf{e}_4^T + \mathbf{e}_4\,\mathbf{e}_2^T + \sum_{\substack{j=1\\ j\neq 2,4}}^{4} \mathbf{e}_j$$

$$= \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix} + \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

$$= \begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&1&0&0\end{bmatrix} + \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}$$

These matrices are a special example of a much larger class of matrices known as permutation matrices. As we will see, we can generate any permutation matrix as the product of a series of transposition matrices.

Definition 3.23: Givens Rotation

A Givens rotation is an $n \times n$ matrix of the form

$$Q(i,k,\theta) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & -s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$. The Givens rotation matrix has a total of $(n+2)$ nonzero entries.

Definition 3.24: Gauss Transform

Let $n, k \in \mathbb{N}$ with $k < n$. Let $\boldsymbol{\tau} \in \mathbb{R}^n$ be a vector whose the first k components are zero. In other words, suppose $\boldsymbol{\tau}$ is in the form

$$\boldsymbol{\tau}^T = \begin{bmatrix} \underbrace{0 \quad \cdots \quad 0}_{k} & \tau_{k+1} & \cdots & \tau_n \end{bmatrix}$$

Then, a *Gauss transformation* is a matrix

$$L_k = I_n - \boldsymbol{\tau}\, \mathbf{e}_k^T$$

We call the vector $\boldsymbol{\tau}$ a *Gauss vector*.

EXAMPLE 3.4.7

Let $n = 5$ and $k = 2$. We define the 5×1 vector $\boldsymbol{\tau}^T = \begin{bmatrix} 0 & 0 & 3 & -2 & 4 \end{bmatrix}$. The corresponding Gauss transformation is given by

$$L_2 = I_5 + \boldsymbol{\tau}\, \mathbf{e}_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 & 1 \end{bmatrix}$$

As we will see, Gauss transformations, Shear Matrices, Dilation matrices and Transposition Matrices are extremely helpful tools to create a matrix description of Gaussian elimination. We will use these matrices heavily in creating full solution sets to linear-systems problems. Each of these matrices can be written as the sum of special matrix units.

Definition 3.25: Transpose of a Matrix

Let $A \in \mathbb{R}^{m\times n}$ be a matrix with real-valued coefficients. The **transpose** of A, denoted by A^T, is defined to be the matrix $A^T \in \mathbb{R}^{n\times m}$ such that the ith row of A^T is the row vector formed by transposing the ith column vector of A.

Theorem 17: Algebraic Properties of Matrix Transposes

Let $A, B \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}$. Then, all of the following are properties of matrix transposes:

i. Two Transposes: $\left((A)^T\right)^T = A$

ii. Transpose of a Sum: $(A+B)^T = A^T + B^T$

iii. Scalars come out of Transposes: $(cA)^T = c\,A^T$

iv. Transpose of a Matrix Product: $(A\cdot B)^T = B^T A^T$

Proof. Let $A, B \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}$. Let's start by establishing property i. about taking two transposes of a matrix. To this end, let

$$C = A^T$$

Since A is of size $m \times n$, we know that C has n rows and m columns. Moreover, we see by the definition of the transpose that $c_{ki} = a_{ik}$ for each $i \in \{1, 2, ..., m\}$ and $k \in \{1, 2, ..., n\}$. Now, setting $D = C^T$ we see $d_{ik} = c_{ki} = a_{ik}$. Thus, we conclude that $D = A$, which is what we wanted to show.

Next, let's consider the $m \times n$ matrix

$$S = A + B$$

We know $s_{ik} = a_{ik} + b_{ik}$ for each $i \in \{1, 2, ..., m\}$ and $k \in \{1, 2, ..., n\}$. Then

$$S^T = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} & \cdots & a_{m1}+b_{m1} \\ a_{12}+b_{12} & a_{22}+b_{22} & \cdots & a_{m2}+b_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}+b_{1n} & a_{2n}+b_{2n} & \cdots & a_{mn}+b_{mn} \end{bmatrix} = A^T + B^T$$

Finally, let's show that scalar multiplication comes through transposes. To this end, consider

$$(cA)^T = \begin{bmatrix} ca_{11} & ca_{21} & \cdots & ca_{m1} \\ ca_{12} & ca_{22} & \cdots & ca_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{1n} & ca_{2n} & \cdots & ca_{mn} \end{bmatrix} = c \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} = c\,A^T$$

□

Lesson 9, Part 2: Matrix Operations- Suggested Problems

For all the problems below, be sure to explicitly state the dimensions of the matrices you use for each model.

1. For $n = 4$, create each of the following matrices
   i. $S_{41}(3)$
   ii. $D_3(-7)$
   iii. P_{14}
2. Finish proofs of algebraic properties for matrix addition and scalar multiplication

