

## 2.5 Linear Combinations and Linear Independence

### Definition 2.12: Linear Combination of Vectors

Let  $m, n \in \mathbb{N}$ . Let  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^m$  and suppose  $x_1, x_2, \dots, x_n \in \mathbb{R}$ . A **linear combination** of vectors  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$  is a vector  $\mathbf{b} \in \mathbb{R}^m$  written in the form

$$\mathbf{b} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n = \sum_{k=1}^n x_k \mathbf{a}_k.$$

A linear combination of vectors  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^m$  is a sum of  $n$  vectors, each of size  $m \times 1$ . The  $k$ th summand is defined as the output of a scalar-vector multiplication problem, where we've multiplied the  $k$ th vector  $\mathbf{a}_k$  by the  $k$ th scaling coefficient  $x_k$ , for  $k = 1, 2, \dots, n$ . Thus, a linear combination of the set of vectors  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^m$  is a single vector in  $\mathbf{b} \in \mathbb{R}^m$  that results from  $n$  scalar-vector multiplication and  $n - 1$  vector additions.

#### EXAMPLE 2.5.1

Computer monitors are electronic devices used to display digital data on personal computers. Each computer display is an array of individual pixels. These arrays have two dimensions including width and height. The pixel width dimension of a computer monitor is the number of pixels along the width of the display. The pixel height dimension is the number of pixels along the height of the monitor. For example, below we illustrate a model of a computer monitor with a width of 10 pixels and a height of 10 pixels.

**Model of a 10 x 10 bitmapped display**

1	1	2	3	4	5	6	7	8	9	10
2	11	12	13	14	15	16	17	18	19	20
3	21	22	23	24	25	26	27	28	29	30
4	31	32	33	34	35	36	37	38	39	40
5	41	42	43	44	45	46	47	48	49	50
6	51	52	53	54	55	56	57	58	59	60
7	61	62	63	64	65	66	67	68	69	70
8	71	72	73	74	75	76	77	78	79	80
9	81	82	83	84	85	86	87	88	89	90
10	91	92	93	94	95	96	97	98	99	100

Horizontal pixel location along width of monitor

Vertical pixel location along height of monitor

In this model, we have numbered each pixel from 1 to 100. This numbering system is known as a bitmap and is used to assign a set of bits in the computers memory system to store color data for each individual pixel on the display. In order

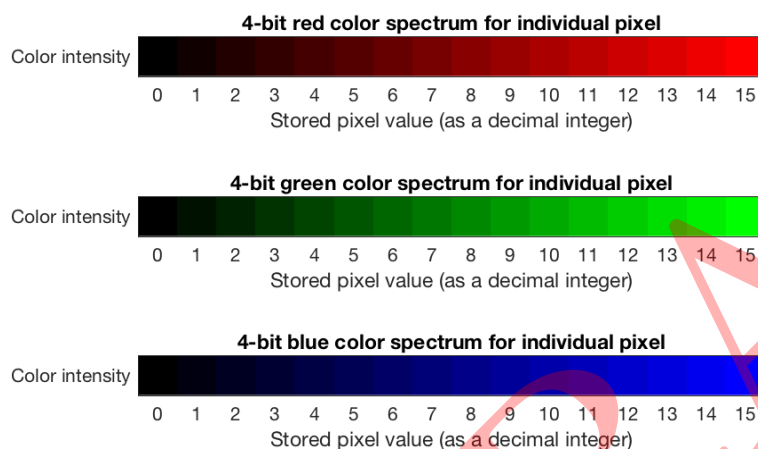
to display color on the screen at each pixel value, your computer's operating system scans the bits stored for each pixel and colors the pixel based on the stored value in each set of bits. The exact color produced for each individual pixel depends on the color model that your computer uses to store color data.

One common color map used to store color in individual pixels is known as the *RGB color model*. In this model, individual pixel colors are created by summing red, green and blue color intensities together. Suppose we store each pixel as a  $3 \times 1$  vector  $\mathbf{x}_k \in \mathbb{R}^3$  given by

$$\mathbf{x}_k = \begin{bmatrix} x_{1k} \\ x_{2k} \\ x_{3k} \end{bmatrix}$$

where the index  $1 \leq k \leq 100$  represents the pixel number as defined by our bitmap.

In this example, we assume each pixel has a 4-bit color depth. In other words, the individual coefficients  $x_{ik}$  of each vector are stored as a 4-bit binary integer in the computer memory, for  $i = 1, 2, 3$ . Because 4-bit binary numbers can range between  $0 \leq x_{ik} \leq 15$ , for  $i = 1, 2, 3$ , each color component in an individual pixel has a total of 16 different intensities as illustrated below.



The final color displayed by each pixel is the sum of the three individual red, green, and blue colors added together.

With this model in mind, we can think of any color displayed on a computer monitor using an RGB color model as a linear combination of the vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

To color the  $k$ th pixel on our monitor, we choose appropriate scaling coefficients  $x_{1k}, x_{2k}, x_{3k}$  and consider the linear combination. Let's create the color orange using the linear combination

$$\begin{bmatrix} 15 \\ 7 \\ 0 \end{bmatrix} = 15 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Notice that each pixel's color intensity is encoded as a linear combination of the colors red, green, and blue.

### EXAMPLE 2.5.2

Suppose we want to model the relationship between the Celsius and Fahrenheit temperature scales. We know that the Celsius scale measures  $0^\circ\text{C}$  for the freezing temperature of water and  $100^\circ\text{C}$  for the boiling point of water. On the other hand, the Fahrenheit scale runs from  $32^\circ\text{F}$  for the freezing point and  $212^\circ\text{F}$  for the boiling point of water. We can create a conversion between these two scales using the following formula

$$f(c) = 32 + 1.8 \cdot c$$

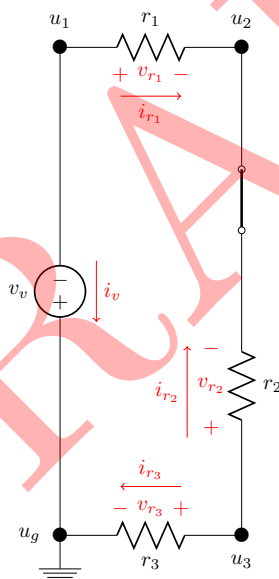
where  $c$  is a given temperature measured in degrees Celsius and  $f(c)$  is the corresponding temperature in degrees Fahrenheit. Using this model, we can convert a number of Celsius measurements to Fahrenheit measurements simultaneously using linear combinations in the form

$$\begin{bmatrix} f(c_1) \\ f(c_2) \\ f(c_3) \\ f(c_4) \end{bmatrix} = 32 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 1.8 \begin{bmatrix} 0 \\ 10 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 32 \\ 50 \\ 68 \\ 86 \end{bmatrix}$$

The Fahrenheit vector is a linear combination of two other related vectors.

### EXAMPLE 2.5.3

Let's recall our model for the electrical behavior of a flashlight from Example ?? . Below we draw the associated idea circuit model:

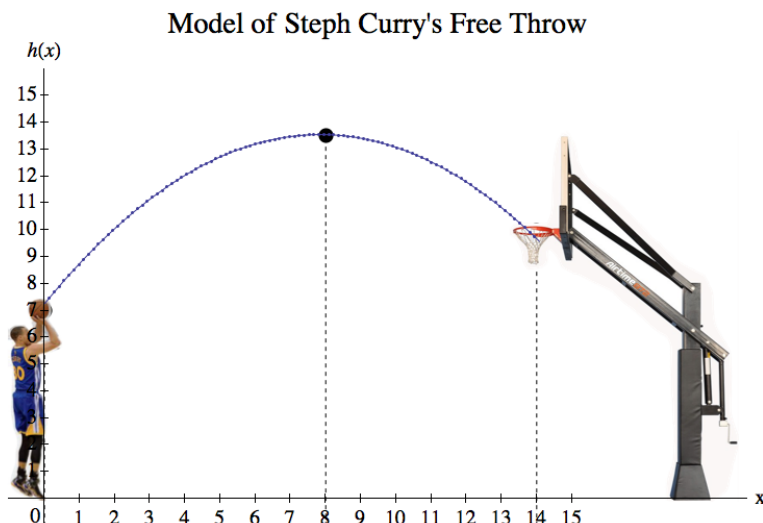


For this circuit, we can find the voltages across each ideal circuit element by calculating the difference between the voltage potentials at the terminal ends of each element. As a linear combination of vectors, this takes the form:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_s \end{bmatrix} = \begin{bmatrix} u_1 - u_2 \\ -u_2 + u_3 \\ u_3 - u_g \\ u_1 - u_g \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + u_g \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

**EXAMPLE 2.5.4**

Stephen Curry plays for the Golden State Warriors as a professional basket ball player in the National Basketball Association (NBA). Suppose we analyze Curry's shooting technique. Below we see a diagram that highlights the typical trajectory of one of Mr. Curry's free-throw shots.



From our study of introductory physics, we choose to model the trajectory of the basket ball using the function

$$h(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2.$$

Suppose the coefficient values of this polynomial are given by

$$a_0 = 7.2500, \quad a_1 = 1.5610, \quad a_2 = -0.0975$$

Then, we can find the set of points in the output given by

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = 7.2500 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1.5610 \cdot \begin{bmatrix} 0 \\ 8 \\ 14 \end{bmatrix} + -0.0975 \cdot \begin{bmatrix} 0 \\ 64 \\ 196 \end{bmatrix} = \begin{bmatrix} 7.250 \\ 13.498 \\ 9.994 \end{bmatrix}$$

**Definition 2.13: Span of a set of vectors**

Let  $m, n \in \mathbb{N}$  and suppose  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^m$ . The **span** of the set of vectors  $\{\mathbf{a}_k\}_{k=1}^n$  is the set of all possible linear combinations of these vectors, denoted as

$$\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\} = \left\{ \sum_{k=1}^n x_k \mathbf{a}_k : x_k \in \mathbb{R} \text{ for } k = 1, 2, \dots, n \right\}.$$

We denote the span of vectors  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$  using angle brackets:

$$\langle \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \rangle = \text{Span}\{\mathbf{a}_k\}_{k=1}^n.$$

**EXAMPLE 2.5.5**

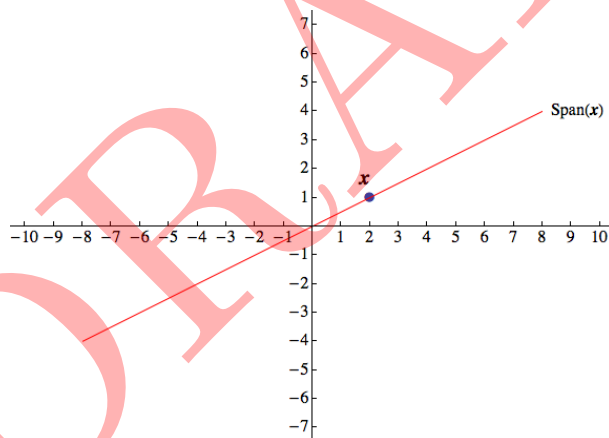
Suppose we are given the vector

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

from Example 2.2.3. Then, the span of this vector is the set

$$\langle \mathbf{x} \rangle = \{ \alpha \mathbf{x} : \alpha \in \mathbb{R} \} = \left\{ \begin{bmatrix} 2\alpha \\ 1\alpha \end{bmatrix} : \alpha \in \mathbb{R} \right\}$$

This is the line given by  $y = 0.5x$  having a slope of 0.5 and y-intercept  $(0, 0)$ . We graph this span below as the red line.



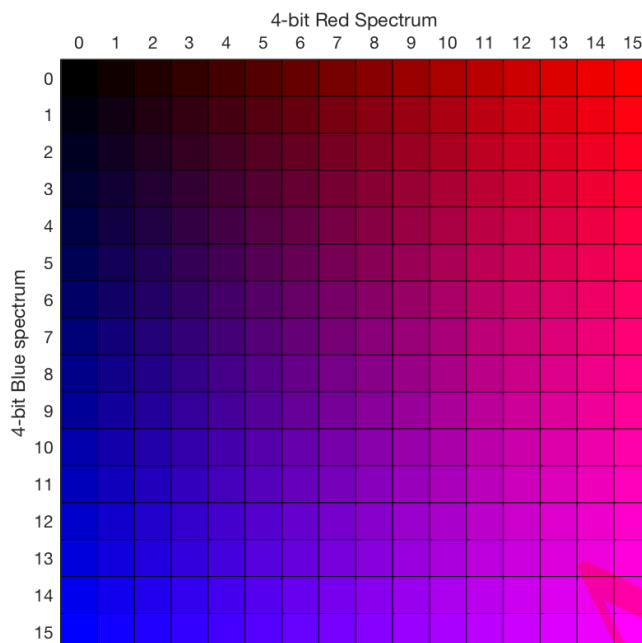
Notice this corresponds with our interpretation of scalar-vector multiplication as a scaling of the vector value.

**EXAMPLE 2.5.6**

Let's consider the RGB color model we discussed above. Suppose we consider

$$\langle \mathbf{a}_1, \mathbf{a}_3 \rangle = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x_1 \\ 0 \\ x_3 \end{bmatrix} \text{ where } x_1, x_3 \in \{0, 1, 2, \dots, 15\} \right\}.$$

This is the set of all colors that can be made by combining red and blue together.



Notice that no greens will appear on this spectrum because the green vector does not contribute to the span. Formally speaking, this set is not actually a span. In particular, we define the span to be the set of all possible linear combinations where the scaling coefficients can be any real number. In this example, the scaling coefficients are integers greater than or equal to 0 and less than or equal to 15. In this case, we are representing the discrete approximation of the span of these vectors in a computer that has a 4-bit rgb color scheme.

**EXAMPLE 2.5.7**

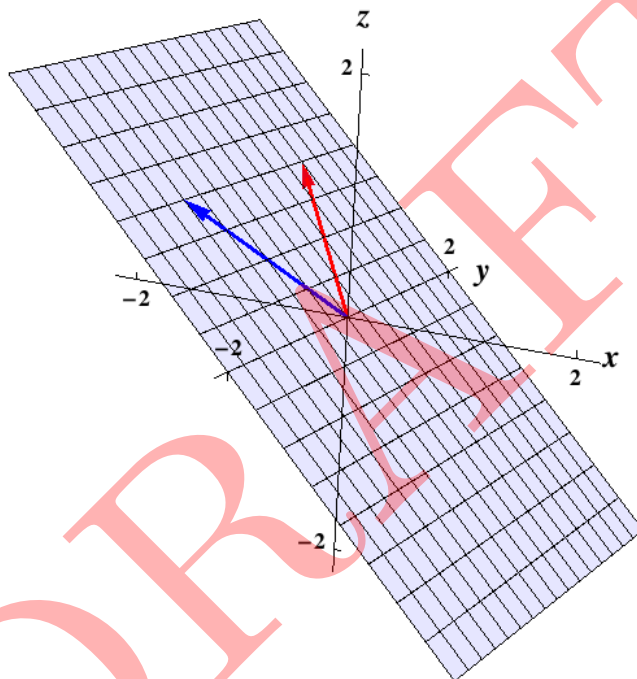
Suppose that we have the two vectors

$$\mathbf{a}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let's create the span of these vectors

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle = \left\{ x_1 \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}$$

We can think of this as the plane in  $\mathbb{R}^3$  through the origin. In the diagram below, we graph the span as the light blue plane. Further, we superimpose our two vectors onto the plane. The vector  $\mathbf{a}_1$  is graphed in red and the vector  $\mathbf{a}_2$  is graphed in blue.



Every point in the light blue plane can be written as a linear combinations of our vectors  $\mathbf{a}_1$  and  $\mathbf{a}_2$ . For example, notice from our figure above, that the “y-axis” lies on our plane. We verify this easily since

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + -\frac{1}{2} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

■

**Definition 2.14: Linearly Dependent Vectors**

Let  $m, n \in \mathbb{N}$ . Suppose  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^m$ . We say that the set of vectors  $\{\mathbf{a}_k\}_{k=1}^n$  is **linearly dependent** if and only one of the vectors can be written as a linear combination of the other vectors. In other words, the set  $\{\mathbf{a}_k\}_{k=1}^n$  is linearly dependent if there exists a  $j \in \mathbb{N}$  with  $1 \leq j \leq n$  and

$$\mathbf{a}_j = \sum_{\substack{k=1 \\ k \neq j}}^n x_k \mathbf{a}_k$$

for some choice  $x_1, x_2, \dots, x_{j-1}, x_{j+1}, \dots, x_n \in \mathbb{R}$ .


**EXAMPLE 2.5.8**

Let's consider the vertex vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

from Example 2.1.1. This set of vectors is linearly dependent because

$$\mathbf{v}_1 = 0 \cdot \mathbf{v}_2 + 0 \cdot \mathbf{v}_3$$


In other words, we can write  $\mathbf{v}_1$  as a linear combination of  $\mathbf{v}_2$  and  $\mathbf{v}_3$ . Indeed, any set of vectors that includes the zero vector will always be linearly dependent. 

**EXAMPLE 2.5.9**

The set of vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -3 \\ 9 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 4 \\ 9 \end{bmatrix}.$$

is linearly dependent. We see this immediately from the fact that

$$\mathbf{a}_1 = -3 \cdot \mathbf{a}_2$$


**Theorem 13**

Any list of  $n$  vectors of size  $m \times 1$  with  $n > m$  is linearly dependent.

In a linearly dependent set of vector, at least one vector can be created from the other vectors in the set via linear combination. As we will see, the concept of linear dependence is intimately connected to the study of the range of the matrix-vector multiplication problem. Given a set of vectors  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^m$ , we can take the span of that set. If we define a matrix  $A$  with  $m$  rows and  $n$  columns, given by

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]$$

then the span of these vectors is the range of the matrix vector multiplication problem. We say that a set of vectors is linearly dependent if we can delete one of the columns of our vector and produce the same range from the reduced matrix.



**EXAMPLE 2.5.10**

The set of vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_4 = \begin{bmatrix} 3 \\ -13 \\ 4 \\ 1 \end{bmatrix}.$$

is linearly dependent. To see this, consider the equation

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -13 \\ 4 \\ 1 \end{bmatrix}$$

We can use the sparsity structure of each vector to see that

$$\mathbf{a}_4 = 3 \cdot \mathbf{a}_1 + 0 \cdot \mathbf{a}_2 - 4 \cdot \mathbf{a}_3$$

**Theorem 14: Test for Linear Dependence**

Let  $m, n \in \mathbb{N}$ . Suppose that  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^m$ . These vectors are linearly dependent if there exists scalars  $x_1, x_2, \dots, x_n$  that are not all zero such that

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n = \mathbf{0}.$$

**Definition 2.15: Linearly Independent Vectors**

Let  $m, n \in \mathbb{N}$ . Suppose that  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^m$  and  $x_1, x_2, \dots, x_n \in \mathbb{R}$ . The set  $\{\mathbf{a}_k\}_{k=1}^n$  of vectors is **linearly independent** if it is NOT linearly independent.

**EXAMPLE 2.5.11**

Consider the vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

from Example 2.5.1. These vectors are linearly independent since we cannot write one of the vectors as a linear combination of the other two. Intuitively, when we state these vectors are linearly independent, we are creating a mathematical formalization of the realization that we cannot combine red and blue to make green.

A linearly independent set of vectors has the property that no vector from the set can be written as a linear combination of the other vectors in the set. Intuitively, we can think of linearly independent sets of vectors as collections of vectors each of which encodes information not found in the other elements in the set. In other words, each vector has information unique to itself when compared with the other vectors in the set.

**Theorem 15: Test for Linear Independence**

Let  $m, n \in \mathbb{N}$ . Suppose that  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}^m$ . These vectors are linearly independent if and only if

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{0}$$

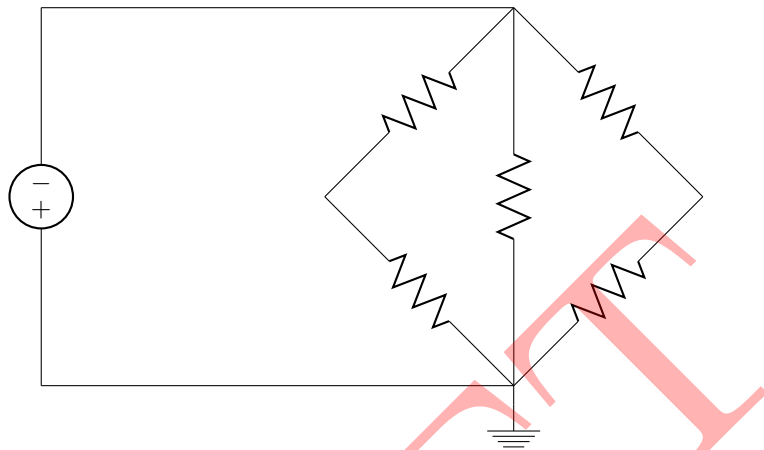
is true only when  $x_k = 0$  for all  $k = 1, 2, \dots, n$ .

## Lesson 6: Vectors and Modeling- Suggested Problems:

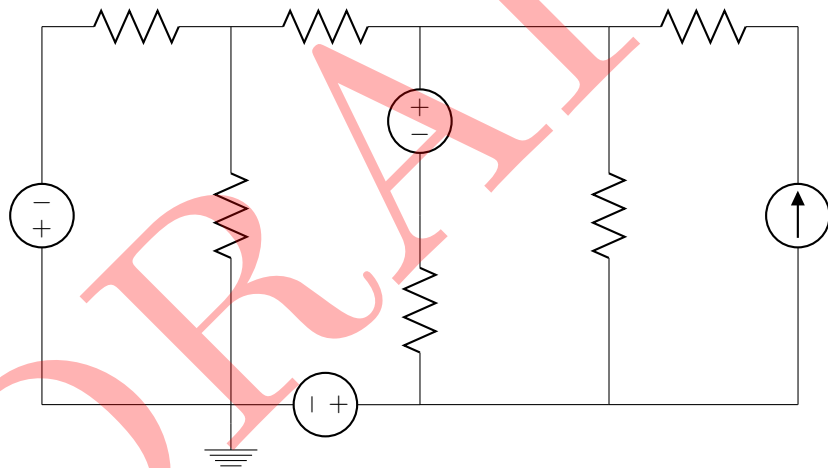
For all the problems below, be sure to explicitly state the dimensions of the vectors you use for each model.

1. Consider the circuits below:

Circuit I:



Circuit II:



- A. Identify each node and label the node potentials. Create a vector  $\mathbf{u}$  to store the node voltage potentials.
- B. Label all voltage and current reference directions.
- C. Introduce vectors  $\mathbf{v}, \mathbf{i}$  for all voltage and current variables in this circuit.
- D. Specifically identify the significance of each entry in these vectors.
- E. Show how to calculate the voltage drop across each element as the difference between node voltage potentials. Write the voltage drop calculations for the entire system as a linear combination of vectors.
- F. Write Kirchhoff's current law (KCL) at each node. Write the KCL equations for the entire system as a linear combination of vectors.

2. Determine whether or not the vector  $\mathbf{b} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$  can be written as a linear combination of the vectors

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

- A. If you can write  $\mathbf{b}$  as a linear combination of  $\mathbf{a}_1, \mathbf{a}_2$ , then specifically identify the scalars  $x_1, x_2 \in \mathbb{R}$  such that  $x_1 \cdot \mathbf{a}_1 + x_2 \cdot \mathbf{a}_2 = \mathbf{b}$ .  
B. If cannot write  $\mathbf{b}$  as a linear combination of  $\mathbf{a}_1, \mathbf{a}_2$ , explain how you know this.
3. Which of the following sets of vectors are linearly independent?

A.  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$                       B.  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$                       C.  $\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

D.  $\begin{bmatrix} 8 \\ -12 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}$                       E.  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}$                       F.  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

4. Which of the following sets of vectors span  $\mathbb{R}^2$ ?

A.  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$                       B.  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$                       C.  $\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

D.  $\begin{bmatrix} 8 \\ -12 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}$                       E.  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix}$                       F.  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$