2.3 Inner Products

Definition 2.9: Inner Product between Vectors

Let $n \in \mathbb{N}$ and suppose that $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then, the inner product between $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{j=1}^{n} x_j y_j.$$

Notice that the dot product is a function with domain $\mathbb{R}^n \times \mathbb{R}^n$ and codomain $\mathbb{R}$. The output of the dot product between any two vectors is a real number.

EXAMPLE 2.3.1

Recall our vector model for storing your performance on all graded assessments in Math 2B from Example 2.1.2. In this model, we stored our individual performance data in the following 4×1 vector

$$\mathbf{g} = \begin{bmatrix} \dfrac{q}{300} \\ \dfrac{e_1}{100} \\ \dfrac{e_2}{100} \\ \dfrac{f}{100} \end{bmatrix}$$

where

$$\begin{aligned} q &= \text{the total number of points you earn on your warm up quizzes} \\ e_1 &= \text{your final score on exam 1} \\ e_2 &= \text{your final score on exam 2} \\ f &= \text{your score on your final exam} \end{aligned}$$

In order to calculate our final grade using vector $\mathbf{g}$ we need to know the grade-category weights assigned to each grade category. In Math 2B, these weights were given in the course syllabus as follows:

I. Warm-Up Quizzes: 10%

II. In-class exam 1: 25%

III. In-class exam 2: 25%

IV. Final Exam: 40%

Let's store these category weights in decimal form using a 4×1 vector

$$\mathbf{c} = \begin{bmatrix} 0.10 \\ 0.25 \\ 0.25 \\ 0.40 \end{bmatrix}$$

DRAFT

With this in mind, we can calculate our final grade percent score (in decimal form) using the following dot product:

$$p_s = \mathbf{g} \cdot \mathbf{c} = 0.10\frac{q}{300} + 0.25\frac{e_1}{100} + 0.25\frac{e_2}{100} + 0.40\frac{f}{100}$$

where p_f is the final percent score you earn in Math 2B and is used to calculate your final grade based on the grade scale included in the course syllabus.

EXAMPLE 2.3.2

Let's suppose that we want to use Riemann integration to find the exact integral of $f(x) = \cos(x)$ on the interval $[0, 2\pi]$. By our study of integral calculus, we know we can evaluate this integral analytically using the definition

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\, h$$

where

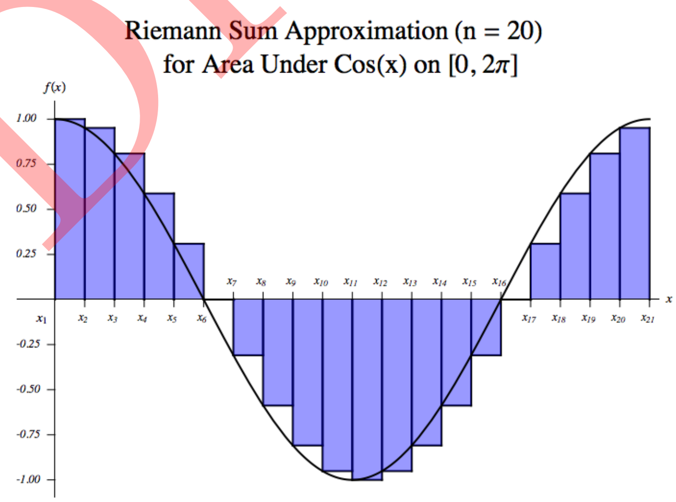
$$h = \frac{b-a}{n}, \qquad \text{and} \qquad x_i = a + i \cdot h$$

Because the cosine function has a closed-form antiderivative (namely $\sin(x)$), we can find our area exactly.

However, for a very wide class of problems, this theoretic definition is not entirely helpful. For any function whose antiderivative cannot be written in terms of elementary functions (e.g. e^{-x^2}), integral calculus does not give us a closed form solution for the integral.

We can instead attempt to numerically approximate the definite integral using Riemann sums. For example, we can choose an approximation scheme in which we discretize our domain space into n equally spaced intervals and sample our function at the discrete endpoints of each interval, just as in Example 2.1.4. If we desire high accuracy, we can use a very fine discretization (which requires more time and energy to compute by the technician).

In this example, we let's discretize our interval $[0, 2\pi]$ at $n = 20$ points. The associated Riemann sum approximation to our integral can be visualized as follows:



In this case, we've used the left-hand rule height of each rectangle touches the graph on the left-hand side. As we recall, there are other methods we can use to approximate definite integrals. Recall from Integral Calculus that a general approach to numerical approximation proceeds as follows:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\,h$$

where $x_i^* \in [x_{i-1}, x_i]$ is any point in the ith subinterval and step size $h = \dfrac{b-a}{n}$ is uniform for each interval.

Notice that our approximation scheme can be written as the dot product

$$\sum_{i=1}^{n} f(x_i^*)\,h = \mathbf{f} \cdot \mathbf{h}$$

where we've create two vectors of dimensions $n \times 1$:

$$\mathbf{f} = \begin{bmatrix} f(x_1^*) \\ f(x_2^*) \\ \vdots \\ f(x_n^*) \end{bmatrix}, \qquad \text{and} \qquad \mathbf{h} = h \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

The choice of the points $x_i^* \in [x_{i-1}, x_i]$ depended on the situation. The three most rudimentary techniques included:

$$\begin{aligned} \text{Left-Hand Rule: } & x_i* = x_{i-1} \\ \text{Right-Hand Rule: } & x_i* = x_i \\ \text{Midpoint Rule: } & x_i* = \frac{x_{i-1} + x_i}{2} \end{aligned}$$

In this application, we have again used the process of **discretization** to transform a problem involving limits into a discrete problem which can be calculated using a finite sum. In this case, we replace integrals with finite sums accomplished by dot products. To improve our approximation, we force the distance between each samples point in the domain to tend toward zero, thus constructing a ideal theoretical tool for evaluating area under the curve.

DRAFT

Theorem 7: Algebraic Properties: Dot Product of Vectors

Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^{n\times 1}$ and $a, b \in \mathbb{R}$. Then, all of the following are algebraic properties of the dot product:

i. Bilinearity:

   a. Linearity in left argument: $(a\mathbf{x} + b\mathbf{y}) \cdot \mathbf{z} = a(\mathbf{x} \cdot \mathbf{z}) + b(\mathbf{y} \cdot \mathbf{z})$

   b. Linearity in right argument: $\mathbf{x} \cdot (a\mathbf{y} + b\mathbf{z}) = a(\mathbf{x} \cdot \mathbf{y}) + b(\mathbf{x} \cdot \mathbf{z})$

ii. Symmetry: $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$

iii. Positivity: $\mathbf{x} \cdot \mathbf{x} > \mathbf{0}$ when $\mathbf{x} \neq 0$ while $\mathbf{0} \cdot \mathbf{0} = 0$.

Below, we will prove part (i) subpart (a) of Theorem 8. The other proofs are left to the reader as an exercise.

Proof. Let $n \in \mathbb{N}$. Suppose $a, b \in \mathbb{R}$ $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. Denote the coefficients of our vectors as follows:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}.$$

Now consider:

$$\begin{aligned} (a\mathbf{x} + b\mathbf{y}) \cdot \mathbf{z} &= \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ \vdots \\ ax_n + by_n \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \\ &= \sum_{j=1}^{n} (ax_j + by_j) z_j \\ &= a \sum_{j=1}^{n} x_j z_j + b \sum_{j=1}^{n} y_j z_j \\ &= a\, \mathbf{x} \cdot \mathbf{z} + b\, \mathbf{y} \cdot \mathbf{z}. \end{aligned}$$

This is exactly what was to be shown. □

The algebraic properties of the inner product come in very handy when we construct solutions to our four fundamental problems in linear algebra. For example, bilinearity of the inner product guarantees that we can interpret matrix-matrix multiplication in many ways. The triangle inequality provides deep intuition in order to construct solutions to the least-square problem.

DRAFT

Lesson 5: Inner Products- Suggested Problems:

1. Derive the Cosine Formula for the Inner Product: Prove Theorems 10, 11, and 12 for yourself (using these notes):
2. Prove Theorem 13: The Cauchy-Schwartz Inequality
3. Use the inner product operation to approximate the area:

$$\int_{-\pi/2}^{\pi/2} \cos(x)dx$$

.

   a. In your approximation scheme, use various values of n
   b. find the exact solution to this problem using Integral Calculus
   c. How fine of a discretization due you need to use to get within 0.1 of the exact answer?

4. Set up an inner product model for your final grade calculation in each class you are currently enrolled in. Write this somewhere very special and refer back to it throughout the quarter.
5. Calculate your GPA using our inner product model. Check to see if Foothill's calculation match your calculations.

DRAFT

2.4 Vector Norms

Definition 2.10: The 2-norm of a vector

Let $n \in \mathbb{N}$ be a positive integer and let $\mathbf{x} \in \mathbb{R}^n$. Then the 2-norm $\mathbf{x} \in \mathbb{R}^n$ is given by

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{j=1}^{n} x_j^2}.$$

We can calculate the $2-$norm of a vector using the associated inner product formula $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.

Remark: Some texts refer to the two-norm as the euclidean norm. From this point forward, let us denote

$$\|\mathbf{x}\| = \|\mathbf{x}\|_2.$$

In general, there are many other vector norms we can consider. However, because the $2-$norm is by far the most powerful from the standpoint of introductory linear algebra, we will focus our attention here.

Using this definition, we can prove a number of interesting facts about the two-norm of a vector, as listed below.

Theorem 8: Algebraic Properties: The $2-$Norm of a Vector

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n \times 1}$ and $a \in \mathbb{R}$. Then, all of the following are algebraic properties of the euclidean norm:

i. Positivity: $\|\mathbf{x}\| \geq 0$ with $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

ii. Homogeneity: $\|a\mathbf{x}\| = |a| \cdot \|\mathbf{x}\|$

iii. Triangle Inequality: $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and suppose $a \in \mathbb{R}$. Let's begin with positivity. Consider

$$\|\mathbf{x}\| \geq 0$$

For any $y \in \mathbb{R}$, recall that if $y \geq 0$, then $\sqrt{y} \geq 0$. This remains true for $y = x_1^2 + x_2^2 + \cdots + x_n^2$. In other words, if we can prove that $\|\mathbf{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$ for all choices of $\mathbf{x}$, we can conclude that $\|\mathbf{x}\| \geq 0$. However, by the positivity property of the inner product, we know $\mathbf{x} \cdot \mathbf{x} \geq 0$. Since $\|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$, we see immediately that $\|\mathbf{x}\|^2 \geq 0$ and we have $\|\mathbf{x}\| \geq 0$. This is what we wanted to show.

Let's continue with the homogeneity property. To this end consider the scalar-vector multiplication

$$a\mathbf{x} = a \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix}$$

DRAFT

With this in mind, we have

$$\begin{aligned}
\|a\mathbf{x}\| &= \sqrt{(ax_1)^2 + (ax_2)^2 + \cdots + (ax_n)^2} \\
&= \sqrt{a^2 \cdot (x_1^2 + x_2^2 + \cdots + x_n^2)} \\
&= \sqrt{a^2} \cdot \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \\
&= |a| \cdot \|\mathbf{x}\|
\end{aligned}$$

This is exactly what we wanted to show.

Finally, we conclude by establishing the triangle inequality for the 2-norm. First, let's recall that the square root function $f(t) = \sqrt{t}$ is increasing so that if $t_1 \leq t_2$, then $\sqrt{t_1} \leq \sqrt{t_2}$. If we can prove

$$t_1 = \|\mathbf{x} + \mathbf{y}\|^2 \leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 = t_2,$$

then we can conclude $\sqrt{t_1} = \|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| = \sqrt{t_2}$, which is the triangle inequality. Consider:

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\
&= \mathbf{x} \cdot (\mathbf{x} + \mathbf{y}) + \mathbf{y} \cdot (\mathbf{x} + \mathbf{y}) \\
&= \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} \\
&= \|\mathbf{x}\|^2 + 2\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2 \\
&= \|\mathbf{x}\|^2 + \sum_{i=1}^{n} 2x_i y_i + \|\mathbf{y}\|^2 \\
&\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2 \\
&= (\|\mathbf{x}\| + \|\mathbf{y}\|)^2
\end{aligned}$$

Notice, the proof above depends heavily on the algebraic properties of the inner product and the inner product formula for the $2-$norm. Also, the second to last expression requires that we know that $\sum_{i=1}^{n} x_i\, y_i \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$. This is the famous Cauchy-Schwarz Inequality and follows from the cosine formula for the inner product that we discuss below. □

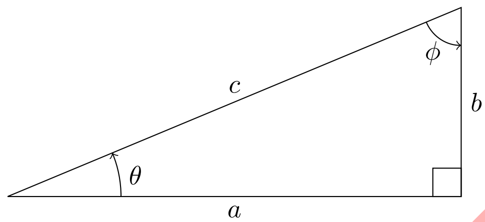
In addition to these algebraic statements, we can also greatly benefit from the study of a geometric property of the inner product. In our discussion of geometry, we will need a few background results including the pythagorean theorem and the law of cosines.

DRAFT

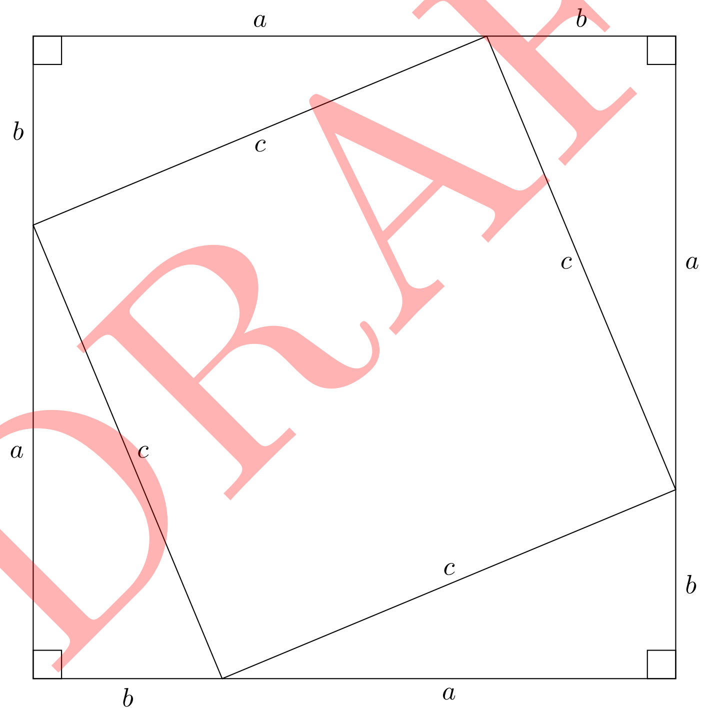
Theorem 9: Pythagorean Theorem

Let a, b, c be positive real numbers representing the length of the base, height and hypotenuse, respectively, of a right triangle. Then $a^2+b^2=c^2$.

Proof. Let's begin by visualizing our right triangle and labeling the length of side as indicated in the theorem statement. Further, let's introduce variables θ and ϕ to represent the two acute angles of our triangle as detailed below:



We know the sum of all three interior angles of our triangle add to $180° = 90° + \theta + \phi$. Using four copies of this triangle, let's construct a special quadrilateral.



Because we know that $\theta + \phi = 90°$, we can immediately conclude that the quadrilateral defined by the hypotenuses forms a square with area c^2. Moreover, the total area of the square is given by

$$(a+b)^2 = a^2 + 2a\,b + b^2$$

DRAFT

If we take away each of the four triangles and leave only the center square then we know

$$c^2 = (a+b)^2 - 4\left(\frac{1}{2}ab\right)$$

$$= a^2 + 2ab + b^2 - \frac{4}{2}ab$$

$$= a^2 + b^2$$

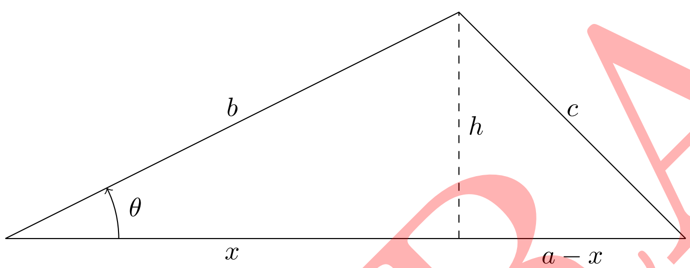
We conclude that $a^2 + b^2 = c^2$ as was to be shown. □

Theorem 10: Law of Cosines

Let a, b, c be positive real numbers representing the length of the three sides of any triangle. Let θ be the angle opposite the side of length c and between the sides of length a and b. Then

$$c^2 = a^2 + b^2 - 2a\,b\cos(\theta)$$

Proof. Let's break the theorem statement into two cases:
Case I: The Acute Case $\left(0 < \theta < \frac{\pi}{2}\right)$



In this case we have $x^2 + h^2 = b^2$. Further we see

$$c^2 = (a-x)^2 + h^2$$

$$= a^2 - 2ax + x^2 + h^2$$

$$= a^2 - 2ax + b^2$$

Since $x = b\cos(\theta)$ by the definition of cosine as the ratio of the adjacent angle over the length of the hypotenuse, we see

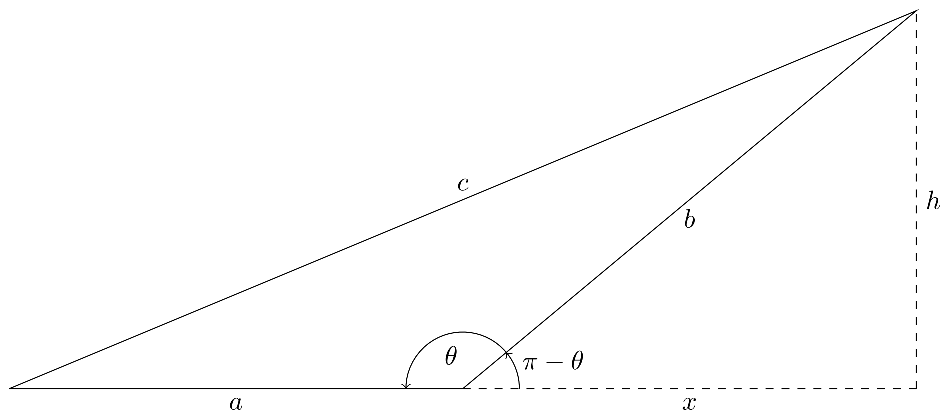
$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$

which is what we wanted to show.

Case II: The Obtuse Case $\left(\frac{\pi}{2} < \theta < \pi\right)$

Below we draw the relevant image for Case II.

DRAFT



In this case, we know that $x^2 + h^2 = b^2$ and $x = b\cos(\pi - \theta)$. Further, by the pythagorean theorem proved above, we know that

$$c^2 = (a + x)^2 + h^2$$

$$= a^2 + 2ax + x^2 + h^2$$

$$= a^2 + b^2 + 2ax$$

We can rewrite our equation $x = b\cos(\pi - \theta) = -b\cos(\theta)$ since the function $\cos(x)$ has a period of 2π. Then we conclude

$$c^2 = (a + x)^2 + h^2$$

$$= a^2 + 2ax + x^2 + h^2$$

$$= a^2 + b^2 - 2ab\cos(\theta)$$

This is exactly what we wanted to prove for Case I.

□

This is a generalization of the pythagorean theorem for any triangle. In the case of the pythagorean theorem, we have that $\theta = 90°$

DRAFT

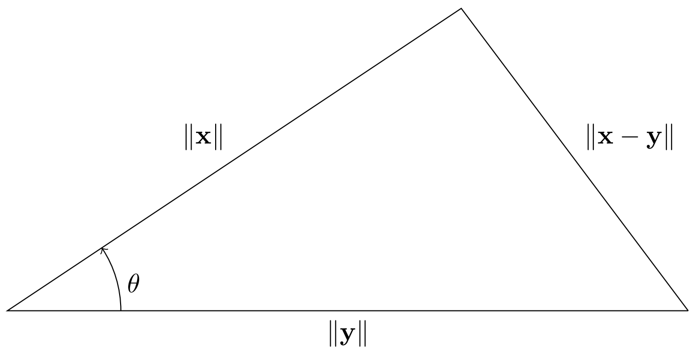
Theorem 11: Cosine Formula for Inner Product

Let $n \in \mathbb{N}$ be a positive integer and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then the dot product satisfies

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos(\theta)$$

where θ is defined to be the angle between vectors $\mathbf{x}$ and $\mathbf{y}$.

Proof. Case I: Assume $\mathbf{x}$ and $\mathbf{y}$ are not scalar multiples of each other. Suppose we begin with two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Consider the triangle defined by these vectors. The length of each side of this triangle can be given by the 2−norm of the vectors:



By the Law of Cosines, we know

$$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\|\mathbf{x}\| \; \|\mathbf{y}\| \; \cos(\theta)$$

Recall, using the algebraic properties of the inner product, we can write

$$\begin{aligned}
\|\mathbf{x} - \mathbf{y}\|^2 &= (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \\
&= \mathbf{x} \cdot (\mathbf{x} - \mathbf{y}) - \mathbf{y} \cdot (\mathbf{x} - \mathbf{y}) \\
&= \mathbf{x} \cdot \mathbf{x} - \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} \\
&= \|\mathbf{x}\|^2 - 2\;\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2
\end{aligned}$$

With this we see

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\|\mathbf{x}\| \; \|\mathbf{y}\| \; \cos(\theta) = \|\mathbf{x}\|^2 - 2\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2$$

By canceling out the appropriate terms using our knowledge of arithmetic, we see

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \; \|\mathbf{y}\| \; \cos(\theta).$$

Case II: Assume $\mathbf{x}$ and $\mathbf{y}$ are scalar multiples of each other (i.e. $\mathbf{y} = a\mathbf{x}$). In this case we know that the angle between our vectors is either $\theta = 0$ or $\theta = \pi$. If $\theta = 0$,

DRAFT

then $a > 0$ and $\cos(\theta) = 1$. On the other hand if $\theta = \pi$, then $a < 0$ and $\cos(\theta) = -1$. In either case, we see

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{y} &= \mathbf{x} \cdot (a\,\mathbf{y}) \\
&= a\,\mathbf{x} \cdot \mathbf{x} \\
&= a\|\mathbf{x}\|^2 \\
&= a\|\mathbf{x}\|\,\|\mathbf{x}\|
\end{aligned}$$

Recall that the sign function $f(x) = \operatorname{sgn}(x)$ is a piecewise function defined as follows:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Then, for any scalar $a \in \mathbb{R}$, we can write $a = \operatorname{sgn}(a)\,|a|$. Moreover, since $|a| = \sqrt{a^2}$, we see

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{y} &= \operatorname{sign}(a)\,\sqrt{a^2}\,\sqrt{\sum_{i=1}^{n} x_i^2}\,\|\mathbf{x}\|_2 \\
&= \operatorname{sign}(a)\,\sqrt{\sum_{i=1}^{n} (ax_i)^2}\,\|\mathbf{x}\|_2 \\
&= \operatorname{sign}(a)\,\|\mathbf{y}\|_2\,\|\mathbf{x}\|_2 \\
&= \cos(\theta)\,\|\mathbf{y}\|_2\,\|\mathbf{x}\|_2
\end{aligned}$$

Thus we see that the cosine formula for the inner product holds □

The cosine formula for the inner product is a powerful tool which will show up repeatedly in many contexts. Notice that the closer the vectors $\mathbf{x}$ and $\mathbf{y}$ are to parallel (the closer θ is to zero), the closer the dot product resembles the norm of the two vectors. In contrast, if $\theta \approx \frac{\pi}{2}$, the dot product is close to zero.

Using this interpretation, we can think of the inner product between vectors as giving a measurement of "parallelity. The larger the magnitude of the inner product between two vectors, the more parallel these vectors are while the smaller the magnitude, the less parallel. Of course, the magnitudes of each vector come into play here, as indicated in the cosine formula.

Orthogonality plays a major role in applied linear algebra and will be the theme of many techniques we develop to solve least-squares problems and linear systems problems. The cosine formula for the dot product gives us a powerful tool to enforce orthogonality between two vectors by guaranteeing that the inner product of two non-zero vector is zero if and only if the vectors are orthogonal.

DRAFT

Now, we can use the law of cosines to make a statement about the relationship between the lengths of general $n \times 1$ vectors $\mathbf{x}, \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$. We can also use the algebraic properties of the dot product to establish the dot product cosine formula stated above.

Theorem 12: Cauchy-Schwartz Inequality

Let $n \in \mathbb{N}$ be a positive integer and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then,

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$$

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then, by the cosine formula for the inner product, we know

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos(\theta)$$

We also know that $-1 \leq \cos(\theta) \leq 1$ for all θ, meaning

$$-\|\mathbf{x}\| \, \|\mathbf{y}\| \leq \mathbf{x} \cdot \mathbf{y} \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$$

Taking the absolute value of the inner expression implies our desired relation. □

Recall that we used the Cauchy-Schwartz Inequality in our verification of the triangle inequality of the 2−norm.

DRAFT

Definition 2.11: Orthogonal vectors

Two vectors are orthogonal if and only if the dot product between these vectors is zero.

EXAMPLE 2.4.1

Using two norms to calculate the similarities in voting records for the US Senate votes.

EXAMPLE 2.4.2

Using inner products between vectors to project one vector onto another.

DRAFT

Lesson 5: Inner Products- Suggested Problems:

1. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^4$ be given by

$$\mathbf{w} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix},$$

   Use these vectors to find each of the following:

   A. $\mathbf{w} \cdot \mathbf{x}$, $\dfrac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{x} \cdot \mathbf{x}}$, and $\dfrac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}$

   B. $\mathbf{y} \cdot \mathbf{z}$, $\dfrac{\mathbf{z} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}$, and $\dfrac{\mathbf{y} \cdot \mathbf{z}}{\mathbf{z} \cdot \mathbf{z}}$

   C. $\dfrac{1}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w}$ and $\dfrac{1}{\mathbf{x} \cdot \mathbf{x}}\mathbf{x}$

   D. $\dfrac{1}{\mathbf{y} \cdot \mathbf{y}}\mathbf{y}$ and $\dfrac{1}{\mathbf{z} \cdot \mathbf{z}}\mathbf{z}$

   E. $\dfrac{\mathbf{w} \cdot \mathbf{x}}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w}$ and $\dfrac{\mathbf{y} \cdot \mathbf{z}}{\mathbf{z} \cdot \mathbf{z}}\mathbf{z}$

   F. $\|\mathbf{w}\|_2$, $\|\mathbf{x}\|_2$, $\|\mathbf{y}\|_2$, and $\|\mathbf{z}\|_2$,

2. Derive the Cosine Formula for the Inner Product: Prove Theorems 10, 11, and 12 for yourself (using these notes):

3. Prove Theorem 13: The Cauchy-Schwartz Inequality

4. Use the inner product operation to approximate the area:

$$\int_{-\pi/2}^{\pi/2} \cos(x)dx$$

   .

   a. In your approximation scheme, use various values of n
   b. find the exact solution to this problem using Integral Calculus
   c. How fine of a discretization due you need to use to get within 0.1 of the exact answer?

5. Set up an inner product model for your final grade calculation in each class you are currently enrolled in. Write this somewhere very special and refer back to it throughout the quarter.

6. Calculate your GPA using our inner product model. Check to see if Foothill's calculation match your calculations.

DRAFT