Chapter 2

Vectors and Vector Operations

In this chapter, we introduce vectors and vector operations. Vectors are central components of all four fundamental problems of applied linear algebra. Just as the properties of real numbers form the building blocks of calculus, the properties of vectors set the foundations for applied linear algebra. We can use vectors and vector operations to do all of the following:

1. **Encode data:** Vectors encode multidimensional data in compact form.
2. **Combine data:** Vector addition and scalar-vector multiplication combine known data to form new, related information that gives insight into the nature of multidimensional data sets.
3. **Understand relationships:** The inner product of two vectors measures the angle between these vectors and indicates the degree to which one vector points in the same direction as another vector.
4. **Measure length and distance:** Vector norms measure the length of a vector and can be used to calculate distances between two vectors.
5. **Study algebraic properties:** By identifying the algebraic properties of vector operations, we enable a theoretic approach to constructing vectors and vector operations.

2.1 Vectors and Vector Modeling

Vectors encode quantitative information. In this chapter, we define vectors as a list of real-valued data entries organized in a vertical or horizontal array. In linear algebra, vectors carry information as multiple dimensional data. The word vector originates from the latin word *advector*, which means carrier. There are two types of vectors we work with in this section: column vectors and row vectors. We begin with the definition of a column vector and illustrate the vast utility of this data structure.

Definition 2.1: Column vector

Let $n \in \mathbb{N}$. A **column vector** $\mathbf{x} \in \mathbb{R}^{n\times 1}$ has n real-valued entries, organized in n rows and one column, in the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

For each $i \in \{1, 2, ..., n\}$, the entry in the ith row of $\mathbf{x}$, labeled x_i, is a real number.

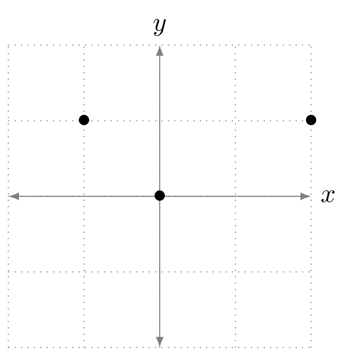
Column vectors are vertical arrays of real numbers. When writing $\mathbf{x} \in \mathbb{R}^{n\times 1}$, the superscript on $\mathbb{R}$ represent the dimensions of vector $\mathbf{x}$. The first argument, n, of the superscript indicates the number of rows of this vector. The second argument, 1, indicates the number of columns of $\mathbf{x}$. The times symbol, $\times$, separates the number of rows of the vector from the number of columns.

We enumerate each entry of a vector using an index variable that we write as a subscript. We label the top most entry of a column vector as x_1 and increase the index variable by one as we move downward through the rows. This enumeration scheme continues until we reach the entry in the last row, which we call x_n.

EXAMPLE 2.1.1

Let's use column vectors to create a vertex model of a triangle in two dimensions. We define our three vertices as follows:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$



In this model, we assume each vector represents a point in two-dimensional space. The entry in the first row of each vertex vector encodes distance along the horizontal axis and the entry in the second row encodes distance along the vertical axis. When we graph these vertices, we do not connect the dots because this model does not encode links between vertices. In the next chapter, we will enhance this modeling scheme to create wireframe models in two or three dimensions. For now, our vertex model only defines points in the cartesian plane.

EXAMPLE 2.1.2

Many college instructors use a grading scheme relying on multiple dimensional data. In this Math 2B course, your final grade depends the four grade categories including warm-up quizzes, exam 1, exam 2 and the final exam. The score you earn in each grade category is computed as a ratio:

$$\text{Grade category score} = \frac{\text{total points you earn in category}}{\text{total points possible in category}}.$$

In Math 2B, there are about 300 points available on your warm-up quizzes. Each exam is worth 100 points. We can use this information to create a 4×1 vector that stores your entire performance in this class:

$$\mathbf{g} = \begin{bmatrix} \dfrac{q}{300} \\ \dfrac{e_1}{100} \\ \dfrac{e_2}{100} \\ \dfrac{f}{100} \end{bmatrix}$$

In this case, we assume

$$\begin{aligned} q &= \text{the total number of points you earn on your warm-up quizzes} \\ e_1 &= \text{your final score on exam 1} \\ e_2 &= \text{your final score on exam 2} \\ f &= \text{your score on your final exam} \end{aligned}$$

As we will see in our discussion of inner products, we use this grade vector along with the vector of grade-category weights to calculate your final grade.

EXAMPLE 2.1.3

The voting record of the Senate, Congress or Supreme Court of the United States of America is openly accessible to be viewed by US citizens. On any particular bill, we can encode the voting record as a column vector. For example, consider the Supreme Court ruling on Obergefell v. Hodges, docket number 14-556. This case concerned same-sex marriage as a fundamental right guaranteed by the Fourteenth Amendment of the US Constitution. This case was decided on June 26, 2015 in favor of same sex marriage. We let -1 stand for a vote against same sex marriage as a right protected by the constitution and 1 represent a vote that supports same sex marriage. With this model in mind, we encode the voting record of the court Justices as follows:

Justice (last name)	Vote on 14-566
Alito	-1
Breyer	1
Gingsburg	1
Kagan	1
Kennedy	1
Roberts	-1
Scalia	-1
Sotomayor	1
Thomas	-1

Just as in the actual case, our model indicates that five court Justices ruled in favor of the argument that same-sex marriage is a fundamental right protected by the US Constitution while four Justices were in opposition.

Definition 2.2: The ith coefficient of a column vector

Let $\mathbf{x} \in \mathbb{R}^{n\times 1}$ be a column vector, given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The real number $x_i \in \mathbb{R}$ in the ith row of $\mathbf{x}$ is called the ith coefficient of vector $\mathbf{x}$ for $i \in \{1, 2, ..., n\}$. We refer to the ith coefficient of $\mathbf{x}$ using the following notation:

i. Subscript notation: x_i

ii. Entry operator notation: $\text{entry}_i(\mathbf{x})$ returns the ith entry of $\mathbf{x} \in \mathbb{R}^n$.

iii. Matrix index notation: $x(i, 1)$ refers to the entry in the ith row and 1st column

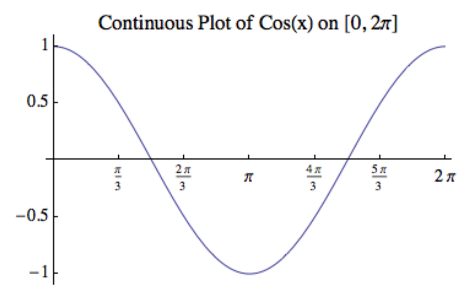
In this textbook, we use subscript notation and the matrix index notation interchangeably. Subscript notation is compact and straight forward. On the other hand, we use matrix index notation exclusively to code in MATLAB.

Definition 2.3: $\mathbb{R}^n$

$\mathbb{R}^{n\times 1}$ is the set of $n \times 1$ real-valued column vectors. Because we often work with column vectors, we use the short hand notation $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. In set builder notation, we have

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R} \text{ for all } i \in \{1, 2, ..., n\} \right\}.$$

EXAMPLE 2.1.4



Suppose we want to graph $f(x) = \cos(x)$ on the interval $x \in [0, 2\pi]$. Most graphing software will produce a graph that approximates continuous behavior. These "continuous plots," as seen in the figure to the right, give a visual representation of the behavior of the entire function on the chosen interval.

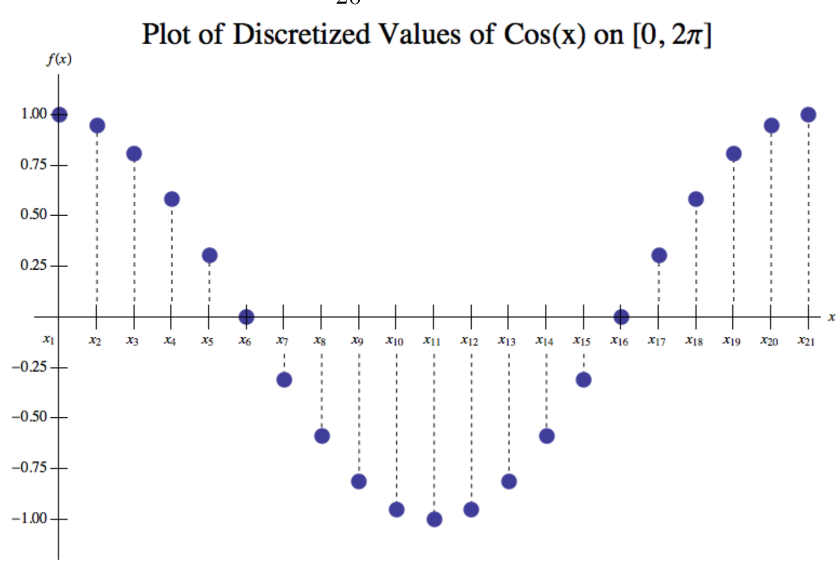
However, it is impossible to plot every point on the graph of $\cos(x)$ over the interval $[0, 2\pi]$ using any computer with finite memory (which is every computer on earth). Instead, graphing programs sample the function $f(x)$ at a sufficient number of data points in the chosen domain and then interpolate these sampled points to produce a graphic for the end user. Let's explore how to sample a function at n points and graph the resulting data.

We divide our given interval into $n+1$ equally-spaced, discrete points. First, let's set $x_1 = 0$ to capture the left endpoint of our interval. We then find a uniform step size $h = \dfrac{2\pi - 0}{n}$ and create each new sample point $x_{i+1} = x_i + h$ where $i = 2, 3, ..., n$. Solving this recursion in terms of x_1, we see $x_{i+1} = x_1 + i \cdot h = i \cdot h$.

Next, we sample the function $f(x)$ at each point x_i and set $y_i = f(x_i) = \cos(x_i)$ for $i = 1, 2, ..., n+1$. With this model, we've created an input vector $\mathbf{x} \in \mathbb{R}^{n+1}$ and output vector $\mathbf{y}\mathbb{R}^{n+1}$ given by

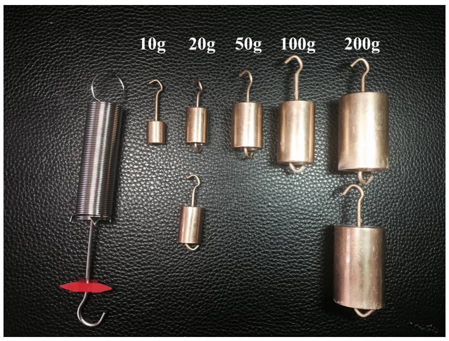
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \cos(x_1) \\ \cos(x_2) \\ \vdots \\ \cos(x_{n+1}) \end{bmatrix}$$

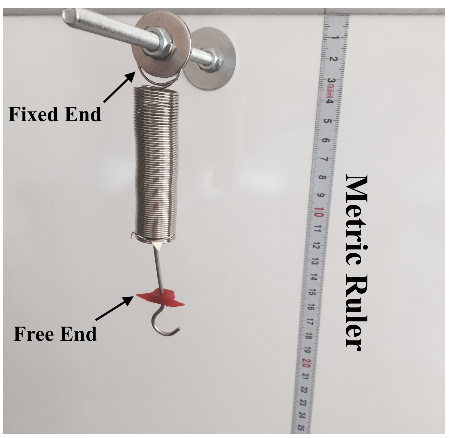
We call this process a **discretization** of the function $f(x)$ on our chosen interval. We graph the relationship between the vector $\mathbf{x}$ and $\mathbf{y}$ with each point on our graph defined by (x_i, y_i). In the graph below, we discretized $\cos(x)$ with $n = 20$ corresponding to a step size of $h = \dfrac{2\pi - 0}{20}$.



The finer we make our discretization (the higher the value of n), the more our graph of sampled points looks like the graph of the continuous function $\cos(x)$ on our given interval.

Through the end of single-variable calculus, we study properties of continuous functions. Graphs of these functions can be extremely useful for visualizing function behavior and studying the properties of families of functions. However, it is impossible to plot all points of a continuous functions exactly. Instead, most graphing processes use some kind of sampling technique to create an approximate plot of the corresponding output. This is our first vivid demonstration of a major theme in applied linear algebra: Continuity is great in theory but in practice we often discretize in order to use the power of computers to solve real-world problems.





EXAMPLE 2.1.5

Let's create an experiment to study the relationship between elongation of a spring and the internal force stored in that spring. To conduct this experiment, we need an extension single spring, an apparatus from which to hang the spring, and a set of masses including two 200g masses. To measure our required data, we need access to a metric ruler (one meter or longer) and a digital scale that measures from 0 to 500 grams in increments of 0.01g.

Let's begin our set up with our extension spring which should have two movable ends connected by a helix. We position our spring vertically so that one of the movable ends is directly above the another. The spring should be perpendicular to the surface of the ground below our feet. We now fix the top end of our spring to our apparatus that holds that top end completely still. We let the bottom end of our spring hang freely.

After we've properly set up our spring, we wait until the free end stops all movement. We say that our spring is in an **equilibrium position** when the free end has a zero velocity and experiences no acceleration. We now attach various masses (measured in kilograms) to the bottom end of our spring and measure the distance from the free end of the spring to the ground below (measured in cm). To store this data, we use two column vectors.

First, we create a 3×1 column vector $\mathbf{m}_{\text{ideal}}$, which stores the ideal masses we use in this experiment. The first coefficient m_1 of vector $\mathbf{m}_{\text{ideal}}$ is set to 0, representing the case where we hang no mass on the movable end of our spring. We now add multiples of 0.2 kg masses to the free end of the spring.

Under this assumption, our ideal mass vector $\mathbf{m}_{\text{ideal}}$ has coefficients that satisfy the recursion $m_1 = 0$ and $m_{i+1} = m_i + 0.2$ where $i \in \{1, 2\}$. In this case, we've limited the range of our indexing variable i because we decided to collect a total of 3 data points. By solving this recursion in terms of m_1, we see $m_{i+1} = 0 + 0.2 \cdot i$ for all i. This set up leads to the vector $\mathbf{m}_{\text{ideal}}$ given below.

Let's use our scale to compare our ideal set up to the actual masses used in our experiment. We create a second vector $\mathbf{m}_{obs}$ which stores the observed masses used in the experiment as follows:

$$\mathbf{m}_{\text{ideal}} = \begin{bmatrix} 0.00 \\ 0.20 \\ 0.40 \end{bmatrix}, \qquad \mathbf{m}_{\text{obs}} = \begin{bmatrix} 0.00000 \\ 0.20010 \\ 0.40031 \end{bmatrix}.$$

We notice that the ideal data vector $\mathbf{m}_{\text{ideal}}$ does not exactly match our observed mass vector $\mathbf{m}_{\text{obs}}$. This is due to errors in the manufacturing process of our masses and perhaps measurement errors from our digital scale. Such errors are an important part of applied mathematical analysis and should not be ignored.

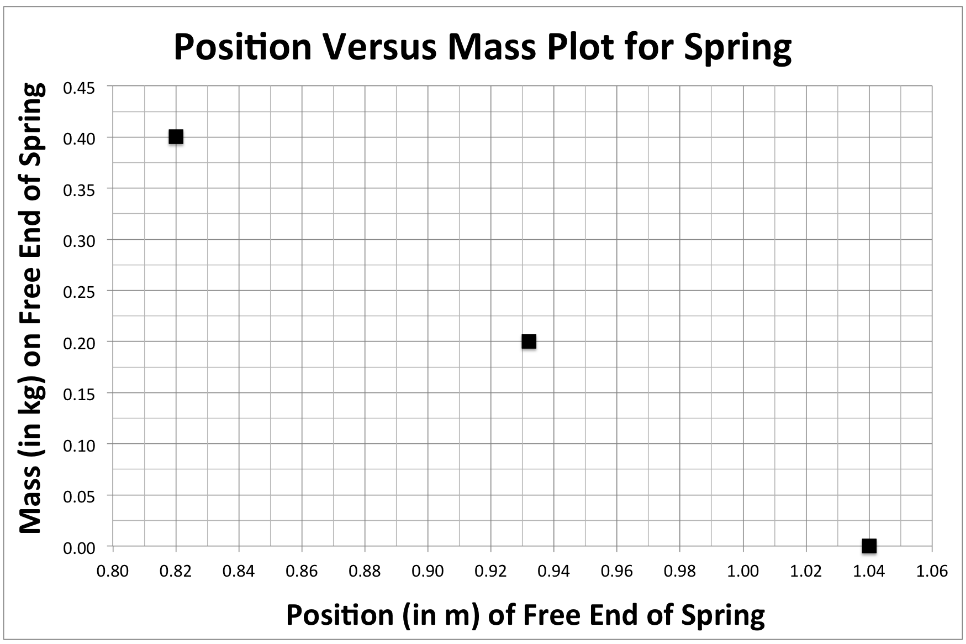
Next, let's record the positions of the movable end of the spring in a position vector $\mathbf{x}_{obs} \in \mathbb{R}^{3\times 1}$. The first coefficient of vector $\mathbf{x}_{\text{obs}}$ is the distance from the free end of the spring at equilibrium with no mass attached to the bottom of the measuring apparatus. Notice that this entry x_1 corresponds the the coefficient

$m_1 = 0$ of $\mathbf{m}_{\text{obs}}$. To find all other coefficients x_i with $i = 2, 3$, we attached our mass m_i to the free end of the spring and wait for the spring to reach an equilibrium position. We then measure the distance from the bottom of the spring to the ground (in m). We enter that measurement into the proper coefficient of $\mathbf{x}_{\text{obs}}$. The row index of the collected position data should be identical to the row index of the entry in $\mathbf{m}_{\text{obs}}$ that stores the numerical value of the mass being used.

For the experiment, we have the following position vector:

$$\mathbf{x}_{\text{obs}} = \begin{bmatrix} 1.040 \\ 0.932 \\ 0.820 \end{bmatrix}$$

where these measurements are written in meters. Below is a graph of a sample data set collected by the author.



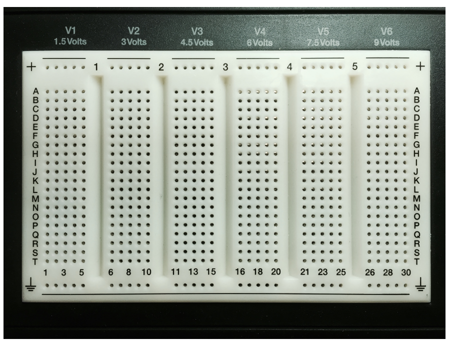
You can access this data set on your course webpage. Each point on the graph is given by (m_i, x_i). When plotting this graph, we see something very interesting. There seems to be a linear relationship between the mass hung on the free end and the distance of the free end to the ground.

EXAMPLE 2.1.6

In this example, we collect empirical data on **Ohm's law**. This is one of key laws of physics that describes the behavior of electric resistors. Ohm's law states that there is a linear relationship between the current flowing through a resistor and the voltage drop across that resistor in an electrical circuit. The associated ideal mathematical model for Ohm's law is given by

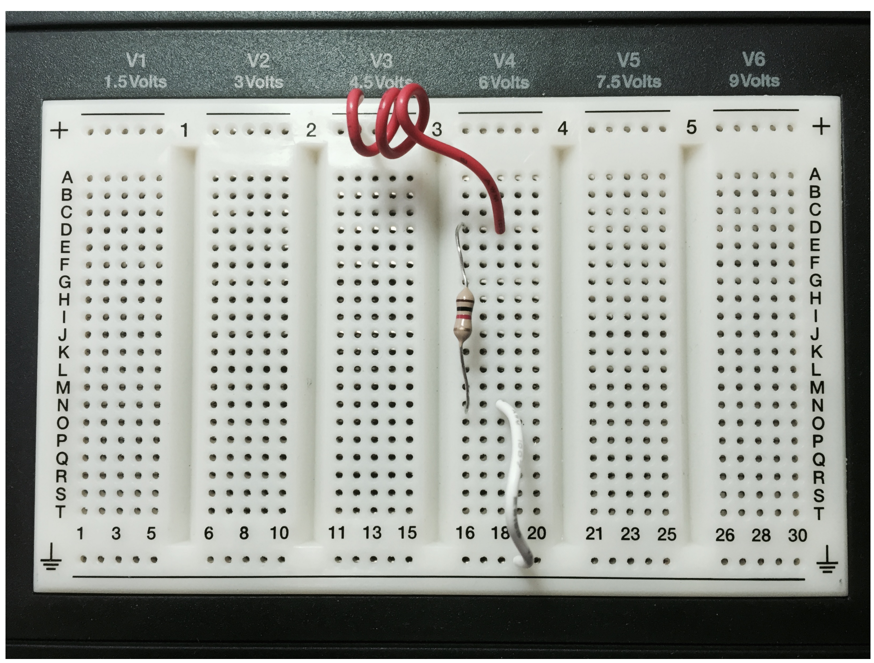
$$V = IR. \tag{2.1}$$

In this equation, V is the voltage drop across the resistor, measured in volts (V). The variable I is the current flowing through the resistor, measured in amperes (A). For a single resistor, the positive constant term R is called the resistance and is measured in ohms (Ω). In ideal circuits with non-variable resistors, resistance measurements are independent of current and voltage measurements.

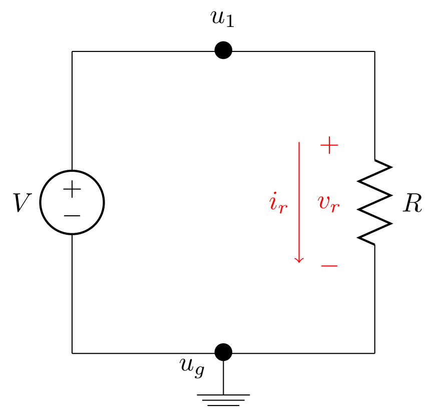


To conduct this experiment, we need a solderless bread board, connecting wires, a resistor, a multimeter and a power source that provides various voltage levels. We begin with our bread board. Each of the points on our bread board is represented by a coordinate system based on row and column labels. We see in the diagram to the left that our bread board's rows are identified by capital letters while columns are indicated by numbers. Moreover, columns are organized in groups of five, each group being separated by a slot. When considering a single row, the five points between two slots are all electrically connected and allow current to flow from any point in that strip. At the top of our board is our power rail which provides voltage ports with varying voltage levels. The 30 connection points on the bottom go to ground, the negative side of our battery input. Ground is also known as the electrical reference node and is assumed to have a voltage value of 0 volts.

To begin our data collection process, we take a single red wire and insert the wire from any of our voltage input to point D18. This will provide a positive voltage potential to all points in row D from point D16 to D20. We input one lead of our resistor into point D16 and input the other lead into point M16. Finally, we take a second white wire and connect point M18 to ground to create a closed circuit.



We can visualize our circuit using an ideal circuit diagram as illustrated below. More about how to create this ideal circuit model can be found in Appendix A: A Primer on Circuit Theory.



Now let's create a 4×1 voltage vector $\mathbf{v}_{\text{ideal}}$ that stores seven ideal voltage levels to be applied to the open end of our red wire. The first coefficient v_1 of vector $\mathbf{v}_{\text{ideal}}$ will have a value of 0 volts, representing the case where we have not turned on our power supply. To create each subsequent entry, we apply multiples of 3V to the free end of the wire. Under this assumption, entries of the vector $\mathbf{v}_{\text{ideal}}$ satisfy the recursion

$$v_1 = 0, \qquad\qquad v_{k+1} = v_k + 3.0$$

where $k \in \{1, 2, 3\}$.

While these ideal voltage levels are useful in designing our experiment, the actual experimental conditions are likely not perfect. To address experimental error, we use our multimeter to measure the actual voltages applied in our experimental set up. We store the observed voltage data in a 4×1 vector $\mathbf{v}_{\text{obs}}$. We then compare the ideal voltages to the observed voltage values as follows:

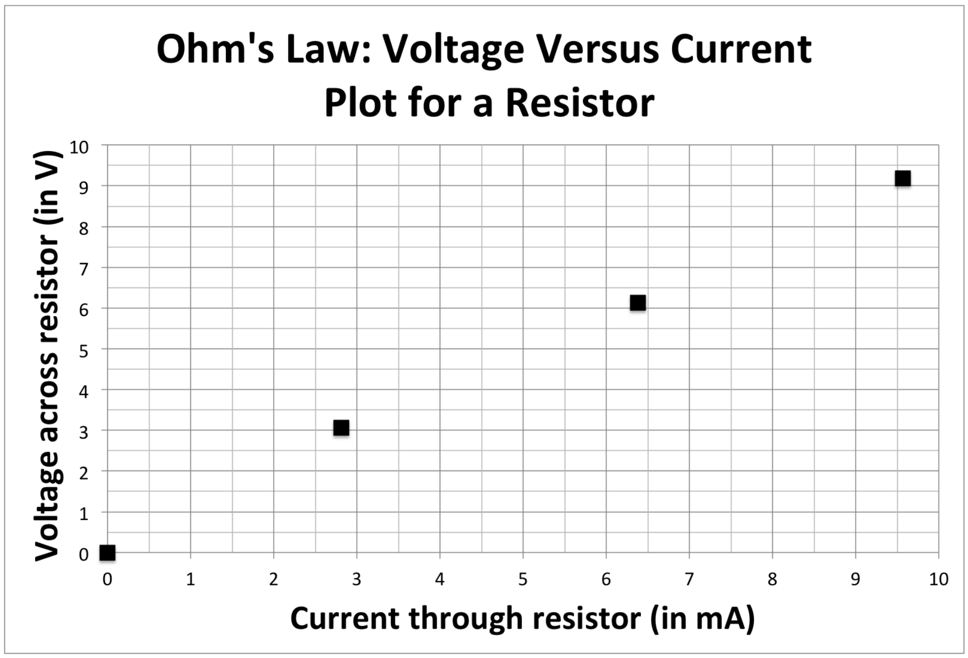
$$\mathbf{v}_{\text{ideal}} = \begin{bmatrix} 0.00 \\ 3.00 \\ 6.00 \\ 9.00 \end{bmatrix}, \qquad\qquad \mathbf{v}_{\text{obs}} = \begin{bmatrix} 0.00 \\ 3.06 \\ 6.13 \\ 9.18 \end{bmatrix}.$$

Discrepancies between the ideal set up and the observed values are due to the variance of voltage output available in our voltage sources (series Duracell 1.5 V batteries) and measurement error from the multimeter.

Next, we create a corresponding 4×1 output vector $\mathbf{i}_{\text{obs}}$, which stores the observed current running through the resistor at any point in time. The first coefficient i_1 of vector $\mathbf{i}$ is the current through the resistor when no voltage is applied across the resistor. Notice that this entry corresponds the the coefficient $v_1 = 0$ of our input vector $\mathbf{v}$. For all other coefficients i_k with $k = 2, 3, 4$, we apply voltage v_k to the free end of the first wire and wait for the circuit to reach an equilibrium position. We then measure the current running through the resistor (in mA) and enter that number into the proper coefficient of $\mathbf{i}$. Notice that the row index of the collected current should be identical to the row index of the entry in $\mathbf{v}$ that stores the numerical value of the voltage being applied. After conducting this experiment, we see

$$\mathbf{i}_{\text{obs}} = \begin{bmatrix} 0.00 \\ 2.81 \\ 6.38 \\ 9.57 \end{bmatrix}$$

Below is a graph of a sample data set collected by the author for the specific resistor pictured in the text above.



You can access this data set on our course webpage. Notice that with this sample data set, we can create an input-output graph. Each point on the graph is given by (v_k, i_k). When plotting this graph, we see something very interesting. Again we see the resemblance of a linear relationship between the voltage across the resistor and the current through the resistor.

One last important note: in this example we do not use our standard row index variable i to represent row indices of each vector. This is is due to the fact that we were using the letter i to represent the coefficient of the current vectors. We also avoid using the index variable j since many electrical engineers state that $j = \sqrt{-1}$. Instead, we use the row index variable k to avoid any confusion. This is an important consideration. When working on applications of linear algebra, we want to be considerate of the professionals working in the scientific field and choose notation that maximizes understanding and minimizes confusion.

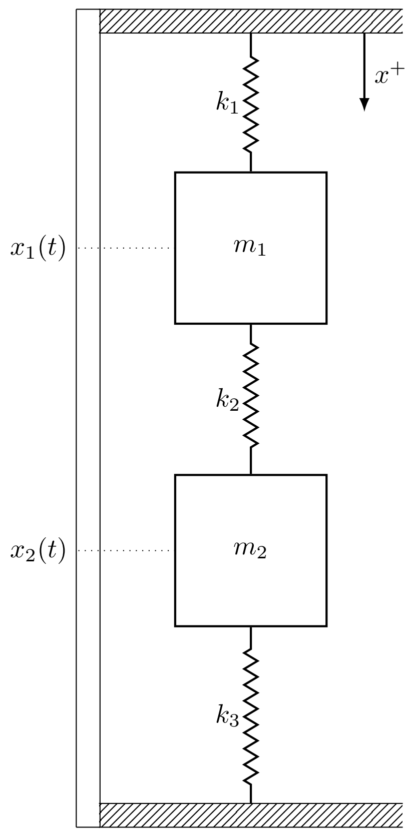
Compare and contrast the example involving the mass spring system and the example involving the resistor. As we will see throughout this text, there are strong analogies between the behavior of mass-spring systems and linear circuits. We will return to this point soon.

Notice also that the last two examples demonstrate reverse processes of Example 2.1.4. Specifically, in Example 2.1.4 we begin with a description of a continuous function $y = \cos(x)$ and use this function description to produce discrete data points that lie on the graph. This example illustrates the process of transforming description of a continuous function into two column vectors. This process can be very helpful for manipulating data on a computer, especially when we begin with knowledge about the behavior of our function.

The reverse process can also be very useful. In particular, it is common to start with collected data and work to produce a continuous function that describes the behavior captured in the data. In Example 2.1.5 and in Example 2.1.6, we do just this. We begin by setting up an experiment and collecting quantitative data. Then, we try to find a mathematical pattern for the behavior of our collected data. In both the Hooke's Law Experiment and the Ohm's Law experiment, we see that our data seems to fit the behavior of the function family of lines. This is where all the work we've done in Calculus pays off. Because we are familiar with families of functions and their properties, we can attempt to guess the type of behavior displayed in our experiment based on our knowledge of the graphs of our data.

EXAMPLE 2.1.7
In Chapter 5 of this text, we will model coupled mass-spring chains using linear systems of equations. For now, we will construct a vector model for a coupled mass-spring chain to encode the positions of our masses at a given point in time. Consider the following diagram representing a coupled mass-spring chain with three springs and two masses.

Spring Mass Chain at equilibrium
with NO external force ($t = 0$ sec)



In the diagram above, let m_i be a given mass (measured in kg) for $i = 1, 2$. In the example we discussed in class, we saw that

$$\mathbf{m} = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 0.20018 \\ 0.10009 \end{bmatrix}$$

Let k_j be the spring constant corresponding to the ith spring, where the first spring is the one on top, the second is right under the first and so on for $j = 1, 2, 3$. In this case, k_j is measured in N/m. For the in-class example, we used the excel data that Jeff collected and the fact that all three springs were made by the same manufacturer to conclude that

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 17.57 \\ 17.57 \\ 17.57 \end{bmatrix}$$

We will encode the positions of individual masses in a vector-valued function $\mathbf{x}(t)$ where

$$\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$$

We will define each component of the vector valued output separately as follows

$$\begin{aligned} x_1(t) &= \text{ the height of mass } m_1 \text{ above ground in meters at time } t \text{ seconds },\\ x_2(t) &= \text{ the height of mass } m_2 \text{ above ground in meters at time } t \text{ seconds }. \end{aligned}$$

We measure distance in the positive direction starting from the zero position on our measuring stick (which is toward the ground) and measuring in the upward direction.

We also assume that at $t \leq 0$, the mass spring system is in a stable equilibrium (no movement) with no external forces acting on the system. Thus $x_i(0)$ is represents the initial position (measured in meters) of the ith mass at $t = 0$ seconds.

We can encode the position of the masses at time $t = 0$ in our mass-spring chain in vector form. Let $\mathbf{x}(0)$ be the vector encoding the position (measured in mm) of the 2 masses at time $t = 0$ seconds. We can physically measure the equilibrium position to find

$$\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.846 \\ 0.434 \end{bmatrix}$$

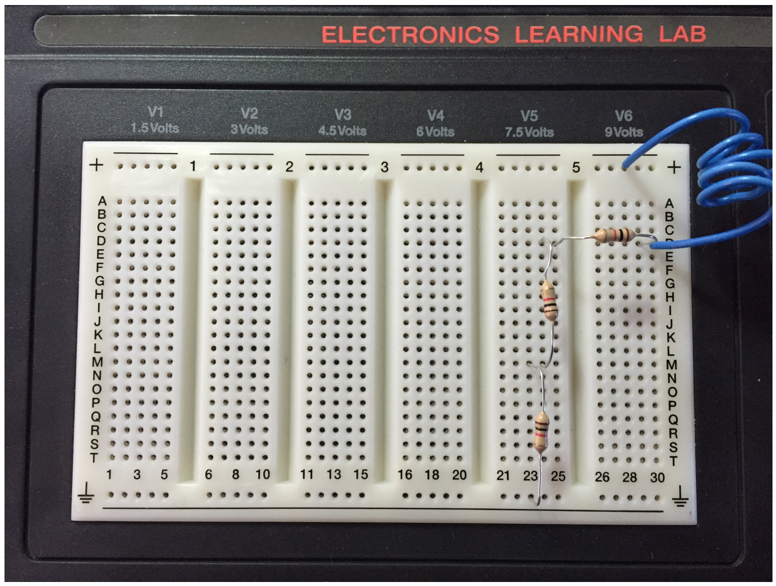
We then introduce the force of gravity on the system by flipping our apparatus 90° so that the white board is perpendicular to the ground. We take measurements of the new positions of our masses when the system settles into equilibrium. Assuming that $t = T$ is the time after which there is no motion in the shifted system, we encode our new positions in vector form as follows:

$$\mathbf{x}(T) = \begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix} = \begin{bmatrix} 0.739 \\ 0.348 \end{bmatrix}$$

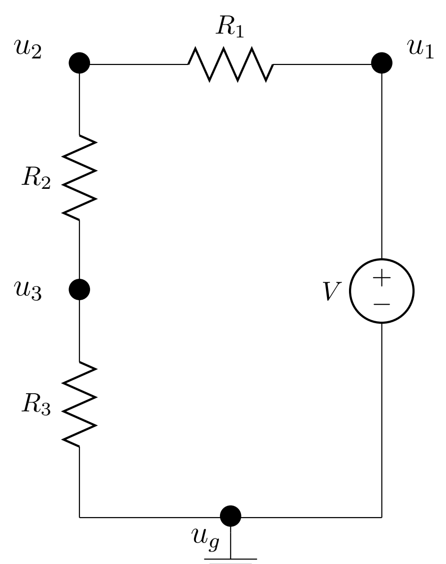
In the next section, we will see how to use operations to calculated the displacement of the masses and the elongation of the individual springs. For now, we have established a useful vector model of a multi-dimensional mechanical system. Notice that in this case the vector model stores positions of masses 1 and 2 at different times in the experiment.

EXAMPLE 2.1.8

In this example, we study the individual node voltages of a circuit with a single voltage source and three resistors connected in series. To conduct this experiment, we will need a solderless bread board, connecting wires, three 1000Ω resistors, a multimeter and a 9V voltage source. We now take a single blue wire and insert the wire from any of our voltage input to point D30. This will provide a positive voltage potential to all points in row D from point D26 to D30. We input one lead of our first resistor R_1 into point D29 and input the other lead to point D25. Our second resistor R_2 should connect points D24 to M24 while the third resistor R_3 connects M23 to ground point G23. We now have an closed circuit.



We can visualize our circuit using an ideal circuit diagram as illustrated below:



To run this experiment, we will measure four node voltage levels, corresponding to the voltage potentials at each of the ground nodes in this circuit. A *node* refers to any point in an electrical circuit where two or more circuit elements are connected. Each node on a circuit is assumed to have one unique voltage level. We will measure the input voltage levels using our multimeter and store the data in a 4×1 vector $\mathbf{u}$. In order to accurately identify our modeling scheme, we introduce some notation to describe our ideal circuit representation by specifically labeling each node of the circuit.

The positive lead of our 9V voltage source is connected to node u_1 while the negative lead is connected to the ground node u_g. Our first resistor $R1$ connects node u_1 to node 2, labeled u_2. Resistor $R2$ connects node u_2 to node u_3. Finally resistor $R3$ connects node u_3 to the ground node u_g.

We now apply our 9V voltage source. To fill in the first coefficient u_1 of vector $\mathbf{u}$, we enter the measured voltage potential at node 1. Similarly, u_2 is the voltage potential measured at node 2 and u_3 is the voltage potential measured at node 3. Finally, u_g stores the reference voltage at the ground node. We can then store this data in vector

$$\mathbf{u} = \begin{bmatrix} 9.15 \\ 6.10 \\ 3.05 \\ 0.00 \end{bmatrix}$$

We will be using this vector modeling scheme again in our study of matrix-vector multiplication to calculate the voltage drop across each element in our circuit. We will also use this model to study both Kirchoff's voltage law and Kirchoff's current law in our discussion of the inner product between vectors.

Look back on our previous examples involving masses, springs, resistors and voltage sources. In both Examples 2.1.5 and 2.1.6, we used vector models to capture the behavior of individual components of a mechanical or electrical system. In particular, by sampling the response of a single component to various loads, we began to detect patterns in our numerical data. This type of observation based inquiry lies at the heart of the scientific method. As we will see, this very powerful modeling scheme lies at the heart of the least squares problem and allows us to describe problems that exist in the physical world by very well chosen mathematical functions.

Compare and contrast the vector models used to create function descriptions for individual components with the vector models in Examples 2.1.7 and 2.1.8. In the latter examples, we store data about the behavior of an entire mechanical or electrical systems in vector form. These models will enable us to describe the state of each component of a system at various points in time. As we will see, we can study the beginning and end behavior of electrical or mechanical systems using both matrix-vector multiplication and linear systems problems. We will study the dynamics of such systems at any time t in our discussion of eigenvalue problems.

Let's look at a completely different use of vector modeling scheme that enables storage of mass quantities of information in a very efficient form.

EXAMPLE 2.1.9

When we search the world wide web, how does google produce a list of relevant resources that match our input query? This is one of many questions from the field of information retrieval which focuses on representing, storing, organizing and accessing informational resources for future reference.

Vectors can be a very powerful tool in web searching. One popular model for encoding a text-based web resource is known as a term-document vector. This is a column vector used to encode the textual information included in a particular document based on the key words, called "terms," that appear in that document. In this example, we create a document library containing a single information resource: a one-sentence "document." We demonstrate how to encode this document as a term-document vector by focusing only on a select few words as "terms." Each term is in bold typeface in the description below. Any word that is not in bold typeface is ignored in this model.

With this in mind, consider the following document:

Document 1: **Applied linear algebra** is a subfield of **mathematics** focused on encoding multidimensional data, manipulating matrices, and solving **linear**-systems problems,**least-squares** problems, and **eigenvalue** problems.

Notice that there are a total of 6 unique terms to be encoded from this document in our library. Because we will be studying this example in more detail later in this text, we choose to create a 10×1 vector with 4 other terms not included in this document.

The rows of this column vector represent all terms from our entire library (in alphabetical order) and the coefficients of each vector represent the frequency with which each term appears in one particular document. Below is term-document vector associated with our library:

Term	Doc 1
algebra	1
applied	1
computer	0
eigenvalue	1
information	0
least squares	1
linear	1
mathematics	1
matrix	0
regression	0

Notice that since document 1 includes the word "applied" one time, there is a 1 in the entry of the vector associated with this term. On the other hand, document 1 does not include the word regression, and thus the corresponding vector has a 0 in that entry.

Using this model, each document in our collection is encoded as a column vector in $\mathbb{R}^{10}$. As we will see in Chapter 3, we can organize these documents into a term-document matrix.As we will see, this model encodes information in such a way that we can use matrix-vector multiplication to determine which of the documents in our library are relevant to our query.

Definition 2.4: Row Vector

Let $n \in \mathbb{N}$ be a positive integer. The vector $\mathbf{x} \in \mathbb{R}^{1\times n}$ given by

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

is a **row vector** with n entries, $x_k \in \mathbb{R}$ for $k \in \{1, 2, ..., n\}$, organized in 1 row and n columns. When referring to a row vector $\mathbf{x} \in \mathbb{R}^{1\times n}$, the superscript on $\mathbb{R}$ represent the dimensions $1 \times n$ of the given vector. Just like when referring to column vectors, the first argument 1 indicates the number of rows and while the second argument n indicated the number of columns. Thus $\mathbb{R}^{1\times n}$ is the set of vectors that have 1 row and n columns.

It is less common to refer specifically to row vectors. Thus, it will always be necessary to write both dimensions when specifically referring to a row vector.

Notice that both column vectors and row vectors are specially organized arrays of real numbers. Traditional calculus education teach vectors as ordered n-tuples of real numbers each entry of which is delimited by a comma. However, in our definition of column and row vectors, each entry of are vectors are written using a very specific format.

Definition 2.5: Equal Column Vectors

Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$ be column vectors These vectors are **equal** if and only if the vectors have the same number of rows (i.e. $n = m$) and $x_i = y_i$ for all $i \in \{1, 2, ..., n\}$. In other words, when we look at the vector equality $\mathbf{x} = \mathbf{y}$, this holds true if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

where the dimensions of the vectors are identical and the individual coefficients are identical.

Lesson 3: Vectors and Modeling- Suggested Problems:

For all the problems below, be sure to explicitly state the dimensions of the vectors you use for each model.

1. Draw a series of diagrams of a spring-mass system used to verify Hooke's Law experimentally. Explain the entire experiment and how the position measurements relate to your diagram. Remember to explain that different masses relate to different observed displacements. Show this by drawing clear diagrams representing a few different cases, as discussed in class. Also, demonstrate how to capture this data in a vector model.

2. Draw a diagram of the ideal circuit used to conduct the Ohm's Law experiment. Explain the entire experiment and how the voltage and current data measurements relate to your diagram. Be sure to discuss the significance of voltages and currents in terms of the circuit. Show how to capture this data in a vector model.

3. Draw the diagram for the spring-mass chain with 3 springs and 2 masses from memory. Explain the entire experiment and how the position measurements relate to your diagram. Be sure to mention the significance of the two measurements in terms of applied forces. Which position data represents the absence of force and which represents the presence of an applied force? Explain your choice of time in teach case.

4. Set up a model for a spring-mass chain with 4 springs and 3 masses. Explain the corresponding vector model to capture the equilibrium positions.

5. Draw a diagram of the ideal circuit corresponding to the series resistor experiment we talked about in class. Explain the entire experiment in detail. Show how to create vectors that store the voltage potential data, the voltage change across each resistor and the current through each resistor. Discuss how this is related to Ohm's Law.

6. Discretize the graph of $y = \sin(x)$ on the interval $[0, 2\pi]$. Use $n = 4, 8, 16$ to produce input and output vectors of various sizes.