## Math 2B: Applied Linear Algebra

True/False For the problems below, circle T if the answer is true and circle F is the answer is false.

1. (T) F Suppose $V_{1}$ and $V_{2}$ are sets. Then any function is a subset of a cross product of sets in the form $V_{1} \times V_{2}$
2. T (F) Any relation from $V_{1}$ to $V_{2}$ is also a function $f: V_{1} \rightarrow V_{2}$.
3. (T) F A function $f: D \rightarrow V_{2}$ is a subset of $D \times V_{2}$
4. (T) F If $f(\mathbf{x})=A \mathbf{x}$, then the $\operatorname{Rng}(f)=\left\{\mathbf{b} \in \mathbb{R}^{m}: \mathbf{b}=A \mathbf{x}\right.$ for some $\left.\mathbf{x} \in \mathbb{R}^{n}\right\}$
5. (T) F If $f(\mathbf{x})=\mathbf{x}^{T}$, then $\operatorname{Dom}(f)=\mathbb{R}^{n}$
6. (T) F Any function is a subset of a cross product of sets
7. (T) F All functions are relations.

Multiple Choice For the problems below, circle the correct response for each question.

1. Which of the following represents the 8 -bit binary representation of the number 207 ?
A. 11110011
B. 11010001
C. 10001011
D. 10111111
E. 11001111
2. Define each of the following sets:

$$
\begin{aligned}
C([0,1]) & =\{f(x) \mid \text { function } f:[0,1] \rightarrow \mathbb{R} \text { is a continuous function on interval }[0,1] .\} \\
C^{(1)}([0,1]) & =\left\{f(x) \mid \text { function } f:[0,1] \rightarrow \mathbb{R} \text { has a continuous first derivative } f^{\prime}(x) \text { on interval }[0,1] .\right\}
\end{aligned}
$$

In other words, $C([0,1])$ is the set of continuous functions $f:[0,1] \rightarrow \mathbb{R}$ while $C^{(1)}([0,1])$ is the set of functions $f:[0,1] \rightarrow \mathbb{R}$ with continuous first derivatives. From Math 1A (Single-Variable, Differential Calculus), we know that if $f(x)$ is differentiable on $[0,1]$, then $f$ is continuous on $[0,1]$. Identify the set theoretic formulation of this theorem: A. $C([0,1]) \subseteq C^{(1)}([0,1])$
B. $C([0,1])=C^{(1)}([0,1])$
C. $C([0,1]) \cap C^{(1)}([0,1])=\emptyset$

$$
\begin{array}{ll}
\text { D. } C^{(1)}([0,1]) \subseteq C([0,1]) & \text { E. } C([0,1]) \cup C^{(1)}([0,1])=C^{(1)}([0,1])
\end{array}
$$

3. Let 10110110 be an 8 -bit binary integer. What is the decimal representation of this number?
A. 109
B. 364
C. 218
D. 80880880
E. 182
4. Let $E \subseteq \mathbb{R} \times \mathbb{R}$ be given by

$$
E=\left\{(x, y): \frac{x^{2}}{1024}+\frac{y^{2}}{729}<1\right\}
$$

Which of the following cannot be true about the relation $E$ ?
A. $\operatorname{Dom}(E)=[-32,32]$
B. $\operatorname{Dom}(E)=(-32,32)$
C. $\operatorname{Rng}(E)=(-27,27)$
D. $E$ is not a function
E. Codomain $(E)=\mathbb{R}$
5. Suppose that we define ellipse $E=\left\{(x, y): \frac{x^{2}}{25}+\frac{y^{2}}{9}=1\right\}$. Find the range, $\operatorname{Rng}(E)$, of this relation.
A. $(-5,5)$
B. $[-5,5]$
C. $[-3,3]$
D. $(-3,3)$
E. $\mathbb{R}$

## Free Response

1. Let $A$ and $B$ be sets. Write the formal set-theoretic definition of a relation from $A$ to $B$. Give two examples of relations that exist in your daily life.

## Solution:

Let $A$ and $B$ be sets. The set $R$ is a relation from $A$ to $B$ iff

$$
R \subseteq A \times B
$$

If $(a, b) \in R$, we write $\mathbf{a}, \mathbf{R} \mathbf{b}$ and say that $a$ is related to $b$.

Recall: The cross product of $A$ and $B$ is the set

$$
A \times B=\{(a, b): a \in A \text { and } b \in B\}
$$

The notation $A \times B$ is read " $A$ cross $B$."
For the examples from life, students might describe:

- Dial-pad relation
- Active-high or active low relation
- Ellipse
- Starbucks cup size relation
- Binary-to-decimal conversion
- Gray-scale relation

2. Let $A$ and $B$ be sets. Suppose $R$ is a relation from $A$ to $B$. Write the formal set-theoretic definition of each of the following:

Solution: Let $A$ and $B$ be sets and let $R$ be a relation from $A$ to $B$ (i.e. suppose $R \subseteq A \times B$ ).

The domain space of relation $R$ is the set $A$.

The domain of the relation $R$ is the set

$$
\operatorname{Dom}(R)=\{x \in A: \text { there is a } y \in B \text { such that }(x, y) \in R\}
$$

The codomain of the relation $R$ is the set

$$
\operatorname{Codom}(R)=B
$$

The range of the relation $R$ is the set

$$
\operatorname{Rng}(R)=\{y \in B: \text { there is a } x \in A \text { such that }(x, y) \in R\}
$$

3. Give an example of a relation in which the domain space is not equal to the domain.

Solution: The touch-tone telephone is well known in the United States. Prior to the existence of Smart Phones, a physical telephone that contain this technology was a ubiquitous part of the US telecommunications infrastructure. The dial pad on many of today's smart phones contains a digital image of the same interface.


Regardless of whether the faceplate is a physical or digital object, each button indicates special properties. The relation implicit in this technology is as follows:

$$
\begin{gathered}
D \subseteq N \times U \\
D=\{(2, \mathrm{~A}),(2, \mathrm{~B}),(2, \mathrm{C}),(3, \mathrm{D}),(3, \mathrm{E}),(3, \mathrm{~F}), \ldots,(9, \mathrm{Y}),(9, \mathrm{Z})\}
\end{gathered}
$$

where $U$ is the set of upper case english letters and $N=\{0,1,2,3,4,5,6,7,8,9\}$. Notice, there are 26 different elements in $R$. Also notice that $(1, \mathrm{G}) \notin D$ and thus D is a proper subset of $N \times U$.
4. Give an example of a relation in which the codomain is not equal to the range.

Solution: Consider the following relation:

$$
E=\left\{(x, y) \in \mathbb{R} \times \mathbb{R}: \frac{x^{2}}{16}+\frac{y^{2}}{625} \leq 1\right\}
$$

If we graph this relation using the Cartesian plane, we see an ellipse. Moreover, we can find

$$
\begin{aligned}
\operatorname{Domain} \operatorname{Space}(E) & =\mathbb{R} \\
\operatorname{Dom}(D) & =[-4,4] \subset \mathbb{R} \\
\operatorname{Codom}(D) & =\mathbb{R} \\
\operatorname{Rng}(D) & =[-25,25] \subset \mathbb{R} .
\end{aligned}
$$

5. State the formal, set-theoretic definition of a mathematical function.

Solution: A function from $A$ to $B$ is a relation $f$ from $A$ to $B$ such that both of the following hold
i. $\operatorname{Dom}(f)=A$
ii. if $(x, y) \in f$ and $(x, z) \in f$, then $y=z$.

We denote the phrase " $f$ is a function from $A$ to $B$ " with the notation $f: A \rightarrow B$. If $B=A$, we say that $f$ is a function on $A$.
6. Convert the following unsigned decimal numbers into the corresponding unsigned binary representation:

$$
\begin{aligned}
47 & =32+8+4+2+1 \\
& =1 \cdot 2^{5}+0 \cdot 2^{4}+1 \cdot 2^{3}+1 \cdot 2^{2}+1 \cdot 2^{1}+1 \cdot 2^{0} \\
& =101111 \\
127 & =64+32+16+8+4+2+1 \\
& =1 \cdot 2^{6}+1 \cdot 2^{5}+1 \cdot 2^{4}+1 \cdot 2^{3}+1 \cdot 2^{2}+1 \cdot 2^{1}+1 \cdot 2^{0} \\
& =1111111 \\
128 & =128 \\
& =1 \cdot 2^{7}+0 \cdot 2^{6}+0 \cdot 2^{5}+0 \cdot 2^{4}+0 \cdot 2^{3}+0 \cdot 2^{2}+0 \cdot 2^{1}+0 \cdot 2^{0} \\
& =10000000
\end{aligned}
$$

7. Convert each of the following unsigned binary numbers into unsigned decimal numbers:

$$
\begin{aligned}
1001101 & =1 \cdot 2^{6}+0 \cdot 2^{5}+0 \cdot 2^{4}+1 \cdot 2^{3}+1 \cdot 2^{2}+0 \cdot 2^{1}+1 \cdot 2^{0} \\
& =64+8+1=77 \\
10111 & =1 \cdot 2^{4}+0 \cdot 2^{3}+1 \cdot 2^{2}+1 \cdot 2^{1}+1 \cdot 2^{0} \\
& =16+4+2+1=23 \\
101011011 & =1 \cdot 2^{8}+0 \cdot 2^{7}+1 \cdot 2^{6}+0 \cdot 2^{5}+1 \cdot 2^{4}+1 \cdot 2^{3}+0 \cdot 2^{2}+1 \cdot 2^{1}+1 \cdot 2^{0} \\
& =256+64+16+8+2+1=347
\end{aligned}
$$

8. Describe the 2-bit gray-scale color map as a function. Define the domain space and codomain of this function. Describe how this relates to digital images.

Solution: When you take a black and white picture with a digital camera, you capture analog information (a collection of fluctuating light waves in the frame of your picture) and transform these waves into digital information (a collection of binary digits inside a computer). The technology inside the camera focuses light from your camera lens onto a digital image sensor. These sensors provide a grid of tiny photosites, each one called a pixel. Each pixel is a light-sensitive electronic device that converts photons from incoming light into an analog voltage level that can then be digitized. Once this digital information is stored, your camera must have a way to convert the digital information of a picture into an image that you can recognize.

In this example, we study the 2-bit gray scale. The gray scale is a standard color model that indicates exactly how binary numbers are translated into shades of gray corresponding to luminous intensities captured at each pixel. To begin our study, we note that for each pixel, our camera has set a color depth (also known as bit depth). This is the number of bits dedicated to each pixel. Below, we show gray scales corresponding to our $2-$ bit depth models.


We can list all of the possible representations of light using our knowledge of binary representations. Once we've done so, we should determine the mapping between each binary number and the corresponding intensity of light. Below is a fictitious example of one way to do this for a black and white image:
In the above diagrams, we've presented the stored binary number in decimal form. However, when these values are stored in a digital camera's memory, these exist as voltage values representing the corresponding binary numbers. A 4-bit gray scale indicates that each pixel has 4 different voltage values used to store the intensity of light sensed at that pixel. For more about this, see the wikipedia articles on "digital camera," "color model," "color depth" and "luminous intensity."

